# Blowup of Smooth Solutions for Relativistic Euler Equations

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**Abstract:** We study the singularity formation of smooth solutions of the relativistic Euler equations in $(3 + 1)$-dimensional spacetime for both finite initial energy and infinite initial energy. For the finite initial energy case, we prove that any smooth solution, with compactly supported non-trivial initial data, blows up in finite time. For the case of infinite initial energy, we first prove the existence, uniqueness and stability of a smooth solution if the initial data is in the subluminal region away from the vacuum. By further assuming the initial data is a smooth compactly supported perturbation around a non-vacuum constant background, we prove the property of finite propagation speed of such a perturbation. The smooth solution is shown to blow up in finite time provided that the radial component of the initial "generalized" momentum is sufficiently large.

## 1. Introduction

In this paper, we study the singularity formation of solutions of the Einstein equations for an isentropic perfect fluid. Due to the hyperbolic nature of these nonlinear equations, one expects singularity formation in the solutions. Indeed, one even expects black holes to form. However, singularity formation in relativistic flow is not yet well-understood; the theory is most lacking in the multi-dimensional case, $(3+1)$-dimensional spacetime.

As a first step in this direction, we consider here the relativistic Euler equations for a perfect fluid in 4-dimensional Minkowski spacetime,

$$Div\ T = 0, \tag{1.1}$$

where

$$T^{ij} = (p + \rho c^2)\mathbf{u}^i\mathbf{u}^j + pg^{ij}, \tag{1.2}$$

is the stress-energy tensor for a perfect fluid, and $g^{ij}$ denotes the flat Minkowski metric, $g^{ij} = diag(-1, 1, 1, 1)$, $\mathbf{x} = (x^0, x^1, x^2, x^3)^T$ with $x^0 = ct$. $\rho$ is the mass-energy

density, $p$ is the pressure, $c$ is the speed of light, and $\mathbf{u}$ is the 4-velocity of the fluid. Recall that since $\mathbf{u} = \frac{1}{c}\frac{d\mathbf{x}}{d\tau}$ ($\tau$ is the proper time, $\mathbf{u}$ is a unit 4-vector in Minkowski space), it follows that

$$(u^0)^2 - \sum_{\alpha=1}^{3}(u^\alpha)^2 = 1,$$

and thus only three of the quantities $u^0$, $u^1$, $u^2$, $u^3$ are independent. We now fix our space-time coordinates as $(t, x^1, x^2, x^3)^T$, set $x = (x^1, x^2, x^3)^T$, $u = (u^1, u^2, u^3)^T$, and let

$$v = \frac{cu}{\sqrt{(1 + |u|^2)}}.$$

One easily derives from Eq. (1.1) the relativistic Euler equations:

$$\begin{cases} \partial_t(\frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2}) + \nabla_x \bullet (\frac{\rho c^2 + p}{c^2 - v^2} v) = 0, \\ \partial_t(\frac{\rho c^2 + p}{c^2 - v^2} v) + \nabla_x \bullet (\frac{\rho c^2 + p}{c^2 - v^2} v \otimes v) + \nabla_x \, p = 0, \end{cases} \tag{1.3}$$

in the unknowns $\rho$, $v$ and $p$. Here $\nabla_x$ denotes the spatial gradient operator. Given a scalar $k$ and 3-vectors $\mathbf{a}$ and $\mathbf{b}$, by the notion $\mathbf{a} \otimes \mathbf{b}$ we mean the matrix $\mathbf{ab}^T$, while

$$\nabla_x \bullet (k\mathbf{ab}^T) = (\nabla_x \bullet (ka_1\mathbf{b}), \nabla_x \bullet (ka_2\mathbf{b}), \nabla_x \bullet (ka_3\mathbf{b}))^T.$$

We consider the Cauchy problem for (1.3) with initial data

$$\rho(x, 0) = \rho_0(x), \, v(x, 0) = v_0(x). \tag{1.4}$$

Equations (1.3) close if we assume an equation of state, $p = p(\rho)$, $p(0) = 0$ with

$$p(\rho) \geq 0, \, 0 < p'(\rho) < c^2, \, p''(\rho) \geq 0, \, for \, \rho \in (\rho_*, \rho^*), \tag{1.5}$$

where $0 \leq \rho_* < \rho^* \leq \infty$. For a $\gamma$-law, $p(\rho) = \sigma^2\rho^\gamma$ with $\gamma \geq 1$, the constant $\rho^*$ is chosen as follows: if $\gamma = 1$, then $\rho^* = \infty$; and if $\gamma > 1$, then $p'(\rho^*) = c^2$. Thus the unknowns for the Cauchy problem (1.3)–(1.4) are $\rho$ and $v$. For more details on the derivation of Eqs. (1.3) and a discussion of (1.5), see [14].

We are interested in the life span of smooth solutions for the Cauchy problem (1.3)–(1.4). For this purpose, we shall discuss two different cases: the case of finite initial energy, and the case of infinite initial energy. For the first case, we shall prove that if the initial data has compact support, then the life span of any non-trivial smooth solution for the Cauchy problem (1.3) and (1.4) is finite. For the second case, we show that if the initial data is a compactly supported perturbation around a non-vacuum background, then the life span of smooth solutions is finite provided that the radial component of the initial "generalized" momentum is sufficiently large; cf. Theorem 3.2.

We start with the infinite energy case. The local existence of classical solutions of the Cauchy problem (1.3)–(1.4) has been established by Makino and Ukai ([6, 7]) provided that the initial data is in the subluminal region away from the vacuum. A sharper result is proved here in Theorem 2.1 in Sect. 2, where the stability of the solution with respect to the initial data (cf. Corollary 2.2) and the properties of finite propagation speed (cf. Lemma 2.3) are presented. In Sect. 3, we first derive some interesting structural properties of (1.3) in Lemma 3.1, then we prove a blowup result for smooth solutions (cf. Theorem

3.2). Our proof is in the spirit of the work of Sideris for classical Euler equations [11] and is based on the largeness of the initial radial component of "generalized" momentum, which of course implies the largeness of the initial velocity. However, in our case, the velocity is still subluminal. In Sect. 4, we prove our blowup result for smooth solutions of (1.3)–(1.4) with non-trivial initial data that has compact support. In Sect. 5, we make some remarks concerning our results. A discussion on the type of singularity is also given. The existence of initial data satisfying our blowup conditions is also shown there. All the results in Sect. 2 are based on the existence of a strictly convex entropy function for (1.3), which was constructed by Makino and Ukai in [6, 7]. For the reader's convenience, we present the construction in the Appendix, correcting a few errors in the original papers.

Before proceeding, we now briefly review the methods and results of singularity formation for nonlinear hyperbolic systems. In one space dimension, the theory is fairly complete. It was proved that a singularity develops in finite time no matter how small and smooth the initial data is; cf. [4, 5, 13]. These results were established by the characteristic method, which is quite powerful in one space dimension. In more than one space dimension, there are no general theorems available mainly because the characteristics become intractable. However, the approach via certain averaged quantities was introduced by Sideris [11] to prove the formation of singularities in three-dimensional compressible fluids. This idea avoids the local analysis of solutions. A similar technique was used to prove other formation of singularity theorems. We refer to [8, 9, 16] for classical fluids, and [2, 10] for relativistic fluids. Blowup results for relativistic Euler equations are announced in [2] and [10]. However, as remarked on p. 154 of [2], " the unpublished proof in [10] contained an error which invalidated the argument ". Furthermore, we note that the coefficient matrices in (2.15) of [2] constructed through (2.16) of [2] are not symmetric away from the equilibrium. But the symmetry of (2.15) in [2] is crucial to prove the finite propagation speed property needed in their proof. Thus the argument in [2] is not complete. Furthermore, we note that the equation of state used in [2] and [10] is different from ours. In addition, the approach of [2] is also different from ours. Our approach is closer to the method of Sideris [11]. Finally, we remark that the equation of state (1.5) in this paper is interesting for cosmology. It includes many physical cases, e.g. $\gamma$-laws, $p(\rho) = \sigma^2 \rho^\gamma$, $\gamma \geq 1$. For instance, the case

$$p(\rho) = \frac{1}{3} c^2 \rho$$

is very important in cosmology; it is the equation of state for the Universe in earliest times after the Big-Bang; see [15]. Some cases discussed in [2] (e.g. when $s = const.$) satisfy (1.5) as well. Another important example (see [15, p.319]) is the equation of state for neutron stars, where

$$p = Ac^5 a(y), \quad \rho = Ac^3 b(y),$$
$$a(y) = \int_0^y \frac{q^4}{\sqrt{1+q^2}} \, dq, \quad b(y) = 3 \int_0^y q^2 \sqrt{1+q^2} \, dq. \qquad (1.6)$$

Here $A$ is a positive constant. This equation of state implies the following asymptotics: $p \to \frac{1}{3} c^2 \rho$ as $\rho \to \infty$ and $p \to \frac{1}{5} A^{2/3} \rho^{5/3}$ as $\rho \to 0$. It is easy to see that

$$p'(\rho) = \frac{c^2 y^2}{3(1+y^2)} > 0, \; p''(\rho) = \frac{2}{9Acy} (1+y^2)^{-5/2} > 0,$$

whenever, $y > 0$. We also note that $y = 0$ is equivalent to $\rho = 0$. Thus the equations (1.6) also satisfy (1.5).

## 2. Existence of Solutions: Infinite Energy Case

In this section, we consider the local existence of smooth solutions for the Cauchy problem (1.3)–(1.4) when the initial data is away from the vacuum. For this purpose, we introduce some convenient notation:

$$\tilde{\rho} = \frac{\rho c^2 + p}{c^2 - v^2},$$

$$\hat{\rho} = \left( \frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2} \right). \tag{2.1}$$

The Cauchy problem (1.3)–(1.4) becomes

$$\begin{cases} \hat{\rho}_t + \nabla_x \bullet (\tilde{\rho} v) = 0, \\ (\tilde{\rho} v)_t + \nabla_x \bullet (\tilde{\rho} v \otimes v) + \nabla_x p(\rho) = 0, \\ \rho(x, 0) = \rho_0(x), \quad v(x, 0) = v_0(x). \end{cases} \tag{2.2}$$

Let $\rho_* < \rho^*$ be non-negative constants in (1.5) subject to the subluminal condition $p'(\rho^*) \leq c^2$. We set

$$z = (\rho, v_1, v_2, v_3)^T$$

and define the region $\Omega_z$ by

$$\Omega_z = \{ z : \rho_* < \rho < \rho^*, v^2 < c^2 \}. \tag{2.3}$$

**Theorem 2.1.** *Assume an equation of state is given as in (1.5). Suppose the initial data $z_0(x) = (\rho_0(x), v_0(x))^T$ is continuously differentiable on $\mathbf{R}^3$, taking values in any compact subset $\mathbf{D}$ of $\Omega_z$ and that $\nabla_x z_0(x) \in H^l(\mathbf{R}^3)$ for some $l > 3/2$. Then there exists $T_\infty, 0 < T_\infty \leq \infty$, and a unique differentiable function $z(x, t) = (\rho(x, t), v(x, t))^T$ on $\mathbf{R}^3 \times [0, T_\infty)$, taking values in $\Omega_z$, which is a classical solution of the Cauchy problem (1.3)–(1.4) on $\mathbf{R}^3 \times [0, T_\infty)$. Furthermore,*

$$\nabla_x z(\cdot, t) \in C^0([0, T_\infty); H^l). \tag{2.4}$$

*The interval $[0, T_\infty)$ is maximal, in the sense that whenever $T_\infty < \infty$,*

$$\lim_{t \to T_\infty} \sup \| \nabla_x z(\cdot, t) \|_{L^\infty} = \infty \tag{2.5}$$

*and/or the range of $z(\cdot, t)$ escapes from every compact subset of $\Omega_z$ as $t \to T_\infty$.*

This theorem will be proved by applying Theorem 5.1.1 in Dafermos [1] for hyperbolic conservation laws endowed with a strictly convex entropy. We state this theorem here for readers convenience.

**Theorem A.** *Assume that the system of conservation laws*

$$U_t + \sum_{\alpha=1}^{m} \partial_{x_\alpha} G_\alpha(U) = 0, \ x \in \mathbf{R}^m, U \in \mathbf{O} \subset \mathbf{R}^n, \tag{*}$$

*is endowed with an entropy $\eta$ with $\nabla^2 \eta(U)$ positive definite, uniformly on a compact subset of $\mathbf{O}$. Suppose the initial data $U(x, 0) = U_0(x)$ is continuously differentiable on $\mathbf{R}^m$, takes values in some compact subset of $\mathbf{O}$ and $\nabla U_0 \in H^l$ for some $l > m/2$. Then there exists $T_\infty$, $0 < T_\infty \leq \infty$, and a unique continuously differentiable function $U$ on $\mathbf{R}^m \times [0, T_\infty)$, taking values in $\mathbf{O}$, which is a classical solution of the initial-value problem (*) with initial data $U_0$ on $[0, T_\infty)$. Furthermore,*

$$\nabla U(\cdot, t) \in C^0([0, T_\infty); H^l).$$

*The interval $[0, T_\infty)$ is maximal, in the sense that whenever $T_\infty < \infty$,*

$$\lim_{t \to T_\infty} \sup \|\nabla U(\cdot, t)\|_{L^\infty} = \infty,$$

*and/or the range of $U(\cdot, t)$ escapes from every compact subset of $\mathbf{O}$ as $t \to T_\infty$.*

*Proof of Theorem 2.1.* We first rewrite (1.3) or (2.2) in the form of conservation laws,

$$\theta_t + \sum_{k=1}^{3} (f^k(\theta))_{x_k} = 0, \tag{2.6}$$

where $\theta = (\theta_0, \theta_1, \theta_2, \theta_3)^T$ and $f^k(\theta) = (\theta_k, f_1^k, f_2^k, f_3^k)^T$ are defined by

$$\begin{aligned} \theta_0 &= \hat{\rho}, \ \theta_j = \tilde{\rho} v_j, \\ f_j^k &= \tilde{\rho} v_j v_k + p \delta_{jk}, \ j = 1, 2, 3. \end{aligned} \tag{2.7}$$

By Theorem A, it is sufficient to show that (2.6) has an entropy $\eta(\theta)$ with $\nabla^2 \eta(\theta)$ positive definite in $\Omega_z$. Such an entropy, due to Makino and Ukai [7], is constructed in the Appendix of this paper.

Define

$$\phi(\rho) = \int_{\rho_m}^{\rho} \frac{c^2}{rc^2 + p(r)} \, dr, \ K = \rho_m c^2 + p(\rho_m), \tag{2.8}$$

$\rho_m$ being any fixed number in $(\rho_*, \rho^*)$. The entropy given in (6.25) below is

$$\eta = c^2 \hat{\rho} - \frac{cK e^{\phi(\rho)}}{\sqrt{c^2 - v^2}}. \tag{2.9}$$

We now verify that $\nabla^2 \eta(\theta)$ is positive definite in $\Omega_z$. To this end, we first compute $\nabla_\theta \eta(\theta)$. By the chain rule, we have

$$w^T = (\nabla_\theta \eta) = (\nabla_z \eta)(\nabla_z \theta)^{-1},$$

where $(\nabla_z \theta)^{-1}$ is defined in (6.10), and $w^T = (w_0, w_1, w_2, w_3)$ is given by

$$
\begin{cases}
w_0 = -\frac{c^3 \Phi(\rho)}{(c^2 - v^2)^{1/2}} + c^2, \\
w_j = \frac{c \Phi(\rho)}{(c^2 - v^2)^{1/2}} v_j, \ j = 1, 2, 3,
\end{cases}
\tag{2.10}
$$

with

$$
\Phi(\rho) = \frac{K e^{\phi(\rho)}}{(\rho c^2 + p)}.
\tag{2.11}
$$

We remark that $w$ can serve as a symmetric variable which reduces (1.3) to a symmetric hyperbolic system [1, 3]. For the Hessian matrix $H$ of $\eta$, we compute

$$
\begin{aligned}
H &= \nabla^2 \eta(\theta) = \nabla_\theta w^T = (\nabla_z w^T)(\nabla_z \theta)^{-1} \\
&= \frac{c \Phi(\rho) E_1}{(\rho c^2 + p)(c^2 - v^2)^{1/2}} H_1 \\
&\equiv \frac{c \Phi(\rho) E_1}{(\rho c^2 + p)(c^2 - v^2)^{1/2}} \begin{pmatrix} A_1 & A_2 v^T \\ A_2 v & A_3 v v^T + A_4 I_3 \end{pmatrix}.
\end{aligned}
\tag{2.12}
$$

Here, $E_1 = \frac{1}{c^4 - p' v^2}$ is given in (6.11) below, and the $A_i$ are given by

$$
\begin{aligned}
A_1 &= c^4(p'c^2 + 2p'v^2 + c^2 v^2), \quad A_2 = -c^2(c^4 + 2c^2 p' + p' v^2), \\
A_3 &= (c^4 + 2c^2 p' + p' v^2 + 2p'(c^2 - v^2)), \quad A_4 = (c^2 - v^2)(c^4 - p' v^2).
\end{aligned}
\tag{2.13}
$$

We now show that $H$ is positive definite. From (2.12), we see that it is sufficient to show $H_1$ is positive definite. Let $\mathbf{r} = (r_0, r^T)^T$ be any 4-vector with $r \in \mathbf{R}^3$. We calculate:

$$
\begin{aligned}
\mathbf{r}^T H_1 \mathbf{r} &= (r_0, r) \begin{pmatrix} A_1 & A_2 v^T \\ A_2 v & A_3 v v^T + A_4 I_3 \end{pmatrix} (r_0, r)^T \\
&= (A_1 r_0^2 + 2A_2 r_0 v^T r + A_3 (v^T r)^2 + A_4 r^2).
\end{aligned}
$$

Letting $\tilde{A}_1 = (1 - \delta) A_1$ with $\frac{1}{2} > \delta > 0$ to be determined in (2.14) below, we have

$$
\begin{aligned}
&(A_1 r_0^2 + 2A_2 r_0 v^T r + A_3 (v^T r)^2 + A_4 r^2) \\
&= \tilde{A}_1 \left( r_0 + \frac{A_2}{\tilde{A}_1} v^T r \right)^2 - \left( \frac{1}{A_1}(A_2^2 - A_1 A_3) + \frac{\delta}{1 - \delta} \frac{A_2^2}{A_1} \right) (v^T r)^2 + \delta A_1 r_0^2 + A_4 r^2 \\
&\geq \left( A_4 - \frac{1}{A_1}(A_2^2 - A_1 A_3) v^2 - \frac{\delta}{1 - \delta} \frac{A_2^2}{A_1} v^2 \right) r^2 + \delta A_1 r_0^2 \\
&\geq \left( \frac{p'(c^2 - v^2)^2(c^4 - p'v^2)}{(p'c^2 + 2p'v^2 + c^2 v^2)} - 2\delta \frac{A_2^2}{A_1} v^2 \right) r^2 + \delta A_1 r_0^2 \\
&\geq \delta A_1 r_0^2 + \delta r^2.
\end{aligned}
$$

Here, we determine $\delta$ by

$$
0 < \delta + 2\delta \frac{A_2^2}{A_1} v^2 < \frac{p'(c^2 - v^2)^2(c^4 - p'v^2)}{(p'c^2 + 2p'v^2 + c^2 v^2)}.
\tag{2.14}
$$

We thus conclude that

$$\mathbf{r}^T H_1 \mathbf{r} \geq (\delta A_1 r_0^2 + \delta r^2).$$

This proves $H_1$ is positive definite in $\Omega_z$. Hence, $H$ is positive definite, and $\eta$ is strictly convex on $\Omega_z$. This completes the proof of Theorem 2.1. $\quad\square$

The existence of a strictly convex entropy guarantees that classical solutions of the initial-value problem depend continuously on the initial data, even within the broader class of admissible bounded weak solutions; see [1]. Here, by admissible bounded weak solution, we mean bounded functions satisfying the initial value problem and entropy inequality in the sense of distributions. The following Theorem B is Theorem 5.2.1 in Dafermos [1]:

**Theorem B.** *Assume that the system of conservation laws* (∗) *is endowed with an entropy* $\eta$ *with* $\nabla^2\eta(U)$ *positive definite, uniformly on compact subset of* **O**. *Suppose U is a classical solution of* (∗) *on* $[0, T)$, *taking values in a convex compact subset N of* **O**, *with initial data* $U_0$. *Let* $\bar{U}$ *be any admissible weak solution of* (∗) *on* $[0, T)$, *taking values in N, with initial data* $\bar{U}_0$. *Then*

$$\int_{|x|<R} |U(x,t) - \bar{U}(x,t)|^2 \, dx \leq ae^{bt} \int_{|x|<R+st} |U_0(x) - \bar{U}_0(x)|^2 \, dx$$

*holds for any* $R > 0$ *and* $t \in [0, T)$, *with positive constants s, a, depending only on N, and a constant b that also depends on the Lipschitz constant of U. In particular,* $\bar{U}$ *is the unique admissible weak solution of* (∗) *with initial data* $\bar{U}_0(x)$ *and values in N.*

The following corollary is a consequence of the convexity of $\eta$ and Theorem B.

**Corollary 2.2.** *Let* $\theta$ *be a classical solution of (2.6) obtained in Theorem 2.1 with initial data* $\theta_0(x)$ *taking values in a compact subset* **D** *of* $\Omega_z$, *and let* $\tilde{\theta}$ *be any admissible weak solution of (2.6) on* $[0, T_\infty)$, *taking values in* **D** *, with initial value* $\tilde{\theta}_0(x) \in$ **D**. *Then*

$$\int_{|x|<R} |\theta(x,t) - \tilde{\theta}(x,t)|^2 \, dx \leq ae^{bt} \int_{|x|<R+st} |\theta_0(x) - \tilde{\theta}_0(x)|^2 \, dx$$

*holds for any* $R > 0$ *and* $t \in [0, T_\infty)$, *with positive constants s, a, depending only on* **D**, *and a constant b that also depends on the Lipschitz constant of* $\theta$. *In particular,* $\tilde{\theta}$ *is the unique admissible weak solution of (2.6) with initial data* $\tilde{\theta}_0(x)$ *and values in* **D**.

In the next lemma, we will show that a compactly supported perturbation around a non-vacuum background propagates with finite speed. For this purpose, we consider the following Cauchy problem

$$\begin{cases} \hat{\rho}_t + \nabla_x \bullet (\tilde{\rho}v) = 0, \\ (\tilde{\rho}v)_t + \nabla_x \bullet (\tilde{\rho}v \otimes v) + \nabla_x p(\rho) = 0, \\ (\rho_0(x) - \bar{\rho}, v_0(x)) = 0, \quad for \ |x| \geq R, \end{cases} \tag{2.15}$$

where, $R > 0$, and $0 < \bar{\rho} \in (\rho_*, \rho^*)$ satisfies the subluminal condition,

$$p'(\rho^*) \leq c^2. \tag{2.16}$$

**Lemma 2.3.** *Let* $(\rho, v)(x, t)$ *be a* $C^1$ *solution of the Cauchy problem (2.15) (equivalent to (1.3) with the same initial data), where the initial data* $(\rho_0, v_0)(x)$ *takes values in a compact subset of* $\Omega_z$ *(cf. (2.3)). Then the support of* $(\rho - \bar\rho, v)(x, t)$ *is contained in the ball* $B(t) = \{x : |x| \le R + st\}$ *where*

$$s = \sqrt{p'(\bar\rho)}$$

*is the sound speed in the far field.*

*Proof.* This lemma is a consequence of the local energy estimates. It will be proved using the method of [11] for symmetric hyperbolic systems. For this purpose, we first observe that $w = (\nabla_\theta \eta)^T$ given in (2.10) (where $\eta$ is as in (2.9)) renders the system (2.15) symmetric hyperbolic [3]:

$$A^0(w)\frac{\partial w}{\partial t} + \sum_{i=1}^{3} A^i(w)\frac{\partial w}{\partial x_i} = 0, \tag{2.17}$$

where the coefficient matrices $A^\alpha(w) = (A^\alpha_{mn})$, $\alpha, m, n = 0, 1, 2, 3$ are given by (6.5); that is

$$A^0 = (\nabla^2_\theta \eta)^{-1}, \quad A^k = (\nabla_\theta f^k)(\nabla^2_\theta \eta)^{-1}.$$

We now compute the explicit form of these matrices. First of all, we have

$$A^0 = (\nabla^2_\theta \eta)^{-1} = \Psi(\rho)\begin{pmatrix} a_1 & a_2 v^T \\ a_2 v & a_3 vv^T + a_4 I_3 \end{pmatrix}, \tag{2.18}$$

where $I_3$ is the $3 \times 3$ identity matrix, and

$$\Psi(\rho) = \frac{1}{K}(\rho c^2 + p)^2 e^{-\phi(\rho)}, \tag{2.19}$$

and

$$a_1 = \frac{c^4 + 3p'v^2}{c^3 p'(c^2 - v^2)^{3/2}}, \quad a_2 = \frac{c^4 + 2p'c^2 + p'v^2}{c^3 p'(c^2 - v^2)^{3/2}},$$
$$a_3 = \frac{c^2 + 3p'}{cp'(c^2 - v^2)^{3/2}}, \quad a_4 = \frac{1}{c(c^2 - v^2)^{1/2}}. \tag{2.20}$$

For $A^k$, $k = 1, 2, 3$, we first compute

$$(\nabla_\theta f^k) = (\nabla_z f^k)(\nabla_z \theta)^{-1}$$
$$= \begin{pmatrix} 0 & e_k^T \\ [c^2(c^2 + v^2)p'E_1 e_k - c^2(c^2 + p')E_1 v_k v] & [-C_4 v e_k^T + v_k I_3] \end{pmatrix}, \tag{2.21}$$

with $e_k = (\delta_{1k}, \delta_{2k}, \delta_{3k})^T$, where $(\nabla_z f^k)$ is given by

$$(\nabla_z f^k) = \begin{pmatrix} B_3 v_k & B_2 v_k v^T + B_4 e_k^T \\ B_3 v_k v + p' e_k & B_2 v_k vv^T + B_4 v e_k^T + B_4 v_k I_3 \end{pmatrix};$$

cf. (6.13) below. Then we have

$$
\begin{aligned}
A^k &= (\nabla_\theta f^k)(\nabla_\theta^2 \eta)^{-1} = (\nabla_\theta f^k) A^0 \\
&= \Psi(\rho) \begin{pmatrix} a_2 v_k & [a_3 v_k v^T + a_4 e_k^T] \\ [a_3 v_k v + a_4 e_k] & [a_3 v_k v v^T + a_4 v_k I_3 + a_4 (e_k v^T + v e_k^T)] \end{pmatrix}.
\end{aligned} \tag{2.22}
$$

It is clear that the matrices $A^k(w)$ are all real symmetric and smooth in $\Omega_z$. Furthermore $A^0(w) = (\nabla^2 \eta)^{-1}$ is positive definite in $\Omega_z$.

Now, we choose $\rho_m = \bar\rho$ subject to (2.16) for convenience. In this setting, the background state in the $w$-variable becomes $\bar w = w(\bar\rho, 0) = 0$. Set

$$
\tilde A^i(w) = (A^0(w))^{-1} A^i(w),
$$

and define

$$
Q(\lambda, \xi) = \lambda I_4 - \sum_{\alpha=1}^3 \xi_\alpha \tilde A^\alpha(0), \tag{2.23}
$$

where $(\lambda, \xi) \in \mathbf{R} \times S^2$ ($S^2$ is the unit 2-sphere). Using the real symmetry of $A^\alpha(0)$, for each $\xi \in S^2$, we see that the characteristic equation

$$
det\, Q(\lambda, \xi) = 0,
$$

has real roots $\lambda_i(\xi)$, $i = 0, 1, 2, 3$, called the characteristic speeds. Let $\bar\lambda$ be the largest absolute value of these characteristic speeds. For any fixed $(x_0, t_0) \in \mathbf{R}^3 \times (0, T_\infty)$, we define the family of cones

$$
C_\tau = \{(s, x) : |x_0 - x| \le \bar\lambda(t_0 - s), \ 0 \le s \le \tau\}, \tag{2.24}
$$

parametrized by $\tau \in [0, t_0)$ and the associated cross sections

$$
E_\sigma = \{(\mu, x) \in C_\tau : \mu = \sigma\}. \tag{2.25}
$$

We introduce the linear partial differential operator

$$
P = A^0(0)\frac{\partial}{\partial t} + \sum_{\alpha=1}^3 A^\alpha(0)\frac{\partial}{\partial x_\alpha}, \tag{2.26}
$$

where again 0 is the background state in the $w$-variable. Equation (2.17) reads

$$
Pw = (A^0(0) - A^0(w))\frac{\partial w}{\partial t} + \sum_{\alpha=1}^3 (A^\alpha(0) - A^\alpha(w))\frac{\partial w}{\partial x_\alpha}. \tag{2.27}
$$

Now we multiply both sides of (2.27) by $2w^T$, to get

$$
\begin{aligned}
&\partial_t[w^T A^0(0)w] + \sum_{\alpha=1}^3 \tfrac{\partial}{\partial x_\alpha}[w^T A^\alpha(0)w] \\
&= 2w^T \left[ (A^0(0) - A^0(w))\tfrac{\partial w}{\partial t} + \sum_{\alpha=1}^3 (A^\alpha(0) - A^\alpha(w))\tfrac{\partial w}{\partial x_\alpha} \right].
\end{aligned} \tag{2.28}
$$

We integrate (2.28) over $C_\tau$ to obtain

$$
\int_{C_\tau} \sum_{\alpha=0}^{3} \partial_\alpha [w^T A^\alpha(0)w](x, \sigma)\, dx d\sigma
$$

$$
= \int_{C_\tau} 2w^T \sum_{\alpha=0}^{3} (A^\alpha(0) - A^\alpha(w)) \partial_\alpha w \tag{2.29}
$$

$$
\leq C \max_{C_\tau} |\nabla w| \int_0^\tau \int_{E_\sigma} |w|^2(x, \sigma)\, dx d\sigma,
$$

where $\partial_0 = \partial_t$, and we have used the mean value theorem in the estimation of the last step. For the left hand side of (2.29), we want to apply the divergence theorem since it is in divergence form. For this purpose, we need to determine the boundary of $C_\tau$ and the associated unit outer normal vector. The boundary of $C_\tau$ consists of three parts: the cap $E_\tau$ with unit outer normal $(1, 0, 0, 0)^T$, the base $E_0$ with unit outer normal $(-1, 0, 0, 0)^T$, and along the surface

$$
R_\tau = \{(\sigma, x) : |x_0 - x| = \bar\lambda(t_0 - \sigma), 0 \leq \sigma \leq \tau\} \tag{2.30}
$$

the unit outer normal vector is

$$
n = \frac{1}{\sqrt{1 + \bar\lambda^2}} (\bar\lambda, -\nu^T)^T, \quad \nu = \frac{(x_0 - x)}{|x - x_0|}. \tag{2.31}
$$

To see (2.31), we note that on $R_\tau$ one has

$$
(\bar\lambda, -\nu^T)^T \bullet ((t_0 - \sigma), (x_0 - x)^T)^T = 0.
$$

We now apply the divergence theorem to the left hand side of (2.29),

$$
\int_{C_\tau} \sum_{\alpha=0}^{3} \partial_\alpha [w^T A^\alpha(0)w](x, \tau)\, dx d\tau
$$

$$
= \int_{E_\tau} (w^T A^0(0)w)(x, \tau)\, dx - \int_{E_0} (w^T A^0(0)w)(x, 0)\, dx \tag{2.32}
$$

$$
+ \frac{1}{\sqrt{\bar\lambda^2 + 1}} \int_0^\tau \int_{\partial E_\sigma} (\bar\lambda w^T A^0(0)w - w^T \sum_{\alpha=1}^{3} \nu_\alpha A^\alpha(0)w)(x, \sigma)\, dS_x d\sigma,
$$

where $dS_x$ denotes the surface element on $\partial E_\sigma$. The third term of (2.32) on the right hand side can be simplified as follows:

$$
(\bar\lambda w^T A^0(0)w - w^T \sum_{\alpha=1}^{3} \nu_\alpha A^\alpha(0)w)
$$

$$
= w^T A^0(0)(\bar\lambda I_4 - \sum_{\alpha=1}^{3} \nu_\alpha \tilde A^\alpha(0))w \tag{2.33}
$$

$$
= w^T A^0(0) Q(\bar\lambda, \nu)w.
$$

We recall that $A^0(0) > 0$ and

$$
A^0(0) Q(\lambda, \nu) = \lambda A^0(0) - \sum_{\alpha=1}^{3} \nu_\alpha A^\alpha(0) \tag{2.34}
$$

is real symmetric. We claim that for any $\nu \in S^2$,

$$A^0(0)Q(\bar{\lambda}, \nu) \geq 0, \tag{2.35}$$

which will be verified at the end of this proof.

Therefore, we conclude from (2.29), (2.32), (2.33) and (2.35) that

$$\int_{E_\tau} (w^T A^0(0)w)(x, \tau) \, dx$$

$$\leq \int_{E_0} (w^T A^0(0)w)(x, 0) \, dx + C_1 \max_{C_\tau} |\nabla w| \int_0^\tau \int_{E_\sigma} |w|^2(x, \sigma) \, dx d\sigma. \tag{2.36}$$

Since $A^0(0) > 0$, there are positive constants $C_2$ and $C_3$ such that

$$\int_{E_\tau} |w|^2(x, \tau) \, dx \leq C_2 \int_{E_0} |w|^2(x, 0) \, dx + C_3 \int_0^\tau \int_{E_\sigma} |w|^2(x, \sigma) \, dx d\sigma,$$

which, by Gronwall's inequality implies that

$$\int_{E_\tau} |w|^2(x, \tau) \, dx \leq C_2 e^{C_3 \tau} (\int_{E_0} |w|^2(x, 0) \, dx). \tag{2.37}$$

Therefore, if $w(x, 0) = 0$ for $|x - x_0| \leq \bar{\lambda} t_0$, then $w(x, \tau) = 0$ for any $\tau \in [0, t_0)$ and $|x - x_0| \leq \bar{\lambda}(t_0 - \tau)$. This implies that, if $w(x, 0) = 0$ for $|x| > R$, then $w(x, t) = 0$ for $|x| > R + \bar{\lambda} t$.

The next step is to verify that $\bar{\lambda} = \sqrt{p'(\bar{\rho})}$. For this purpose, we compute the largest possible characteristic speed at a constant background state. Now we compute the eigenvalues of

$$\sum_{\alpha=1}^3 \xi_\alpha \tilde{A}^\alpha(0), \quad \xi \in S^2. \tag{2.38}$$

Since

$$\tilde{A}^\alpha = (\nabla_\theta^2 \eta)(\nabla_\theta f^\alpha)(\nabla_\theta^2 \eta)^{-1},$$

the matrix in (2.38) is similar to the matrix

$$M(\xi) = \sum_{\alpha=1}^3 \xi_\alpha \nabla_\theta f^\alpha(\bar{\theta}), \ \xi \in S^2, \tag{2.39}$$

where $\bar{\theta} = (\bar{\rho}, 0, 0, 0)^T$ is the background state in the $\theta$-variable. It is easy to compute:

$$\nabla_\theta f^\alpha(\bar{\theta}) = \nabla_z f^\alpha (\nabla_z \theta)^{-1}|_{\rho=\bar{\rho}, v=0}$$
$$= \begin{pmatrix} 0 & e_\alpha^T \\ p'(\bar{\rho})e_\alpha & 0 \end{pmatrix}, \tag{2.40}$$

where $e_i = (\delta_{1i}, \delta_{2i}, \delta_{3i})^T$. Thus, one has

$$M(\xi) = \begin{pmatrix} 0 & \xi^T \\ p'(\bar{\rho})\xi & 0 \end{pmatrix}. \tag{2.41}$$

Now we claim that $M(\xi)$ has rank 2. This can be seen by

$$MM^T = \begin{pmatrix} 1 & 0 \\ 0 & p'(\bar{\rho})^2\xi\xi^T \end{pmatrix},$$

since $rank(\xi\xi^T) = 1$. Thus, $M(\xi)$ has two non-zero eigenvalues and two zero eigenvalues. We need to find all non-zero eigenvalues of $M(\xi)$. We compute

$$\begin{aligned}
0 = \det(M(\xi) - rI_4) &= \det\begin{pmatrix} -r & \xi^T \\ p'(\bar{\rho})\xi & -rI_3 \end{pmatrix} \\
&= \det\begin{pmatrix} 0 & \xi^T \\ (p'(\bar{\rho}) - r^2)\xi & -rI_3 \end{pmatrix} \\
&= (p'(\bar{\rho}) - r^2)\det\begin{pmatrix} 0 & \xi^T \\ \xi & -rI_3 \end{pmatrix},
\end{aligned} \tag{2.42}$$

and this implies $\pm\sqrt{p'(\bar{\rho})}$ are the two distinct non-zero eigenvalues of $M(\xi)$. Therefore, we have

$$\bar{\lambda} = \sqrt{p'(\bar{\rho})}. \tag{2.43}$$

Notice that we did not use (2.35) to obtain (2.43).

The last step is to verify (2.35). Since $\bar{\lambda} = s = \sqrt{p'(\bar{\rho})}$, we have

$$A^0(0) = \frac{K}{c^2}\begin{pmatrix} \frac{1}{s^2} & 0 \\ 0 & I_3 \end{pmatrix}, \quad A^\alpha(0) = \frac{K}{c^2}\begin{pmatrix} 0 & e_\alpha^T \\ e_\alpha & 0 \end{pmatrix}, \quad \alpha = 1, 2, 3,$$

and thus

$$A^0(0)Q(\bar{\lambda}, \nu)) = \frac{K}{c^2}\begin{pmatrix} \frac{1}{s} & -\nu^T \\ -\nu & sI_3 \end{pmatrix}. \tag{2.44}$$

Let $\mathbf{r} = (r_0, r^T)^T$ be any 4-vector with $r \in \mathbf{R}^3$, we compute:

$$\begin{aligned}
\mathbf{r}^T A^0(0)Q(\bar{\lambda}, \nu))\mathbf{r} &= \frac{K}{c^2}(r_0, r^T)\begin{pmatrix} \frac{1}{s} & -\nu^T \\ -\nu & sI_3 \end{pmatrix}(r_0, r^T)^T \\
&= \frac{K}{c^2}(\frac{1}{s}r_0^2 - 2r_0\nu^T r + s|r|^2) \\
&\geq \frac{K}{c^2}(\frac{1}{s}r_0^2 - 2r_0|r| + s|r|^2) \\
&= \frac{K}{sc^2}(r_0^2 - 2sr_0|r| + s^2|r|^2) \\
&= \frac{K}{sc^2}(r_0 - s|r|)^2 \geq 0.
\end{aligned} \tag{2.45}$$

Here, we have used

$$(\nu^T r) \leq \sqrt{(\nu^2)(r^2)} = |r|.$$

Thus, we have proved (2.35). The proof of this lemma is complete.    □

### 3. Singularity Formation: Infinite Energy Case

In this section, we prove the singularity formation of smooth solutions of (2.15) when the initial radial "generalized" momentum is large. To begin, we prove the following two easy but useful identities.

**Lemma 3.1.** $\hat{\rho}$ and $\tilde{\rho}$ satisfy the following identities:

$$\hat{\rho} = \frac{1}{c^2}\tilde{\rho}v^2 + \rho,$$
$$\tilde{\rho} = \frac{1}{c^2}\tilde{\rho}v^2 + (\rho + \frac{p}{c^2}).$$

(3.1)

*Proof.* From (2.1), it is easy to see

$$
\begin{aligned}
\hat{\rho} &= \left[\frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2}\right] \\
&= \frac{\rho c^4 + pc^2 - pc^2 + pv^2}{c^2(c^2 - v^2)} \\
&= \frac{\rho c^2 c^2 + pv^2}{c^2(c^2 - v^2)} \\
&= \frac{\rho c^2 v^2 + \rho c^2(c^2 - v^2) + pv^2}{c^2(c^2 - v^2)} \\
&= \frac{\rho c^2 + p}{c^2(c^2 - v^2)}v^2 + \frac{\rho c^2(c^2 - v^2)}{c^2(c^2 - v^2)} \\
&= \frac{1}{c^2}\tilde{\rho}v^2 + \rho.
\end{aligned}
$$

Hence, $\tilde{\rho} = \hat{\rho} + \frac{p}{c^2}$ implies

$$\tilde{\rho} = \frac{1}{c^2}\tilde{\rho}v^2 + \rho + \frac{p}{c^2}.$$

□

We again denote the sound speed in the far field by

$$s = \sqrt{p'(\bar{\rho})},$$

and define the following quantities:

$$M(t) = \int [\hat{\rho}(\rho, v) - \hat{\rho}(\bar{\rho}, 0)](x, t)\, dx,$$
$$F(t) = \int \tilde{\rho}v \bullet x\, dx.$$

(3.2)

By Lemma 2.3, both $M(t)$ and $F(t)$ are well-defined as long as the smooth solution exists. Using these two quantities, we shall show that the smooth solution of (2.15) obtained in Theorem 2.1 blows up in finite time if the initial data is subject to some restrictions. Roughly speaking, if $M(0) > 0$, and $F(0) > 0$ is sufficiently large, then the solution will blow up in finite time.

**Theorem 3.2.** *Assume that the initial data of (2.15) and $\bar{\rho}$ are chosen such that $M(0) > 0$, $F(0) > 0$ and $s^2 < \frac{1}{3}c^2$. If*

$$F(0) > \Gamma = \frac{32\pi s}{3(1 - \frac{3s^2}{c^2})} R^4 \max \hat{\rho}_0(x),$$

*then the smooth solution of the Cauchy problem (2.15) obtained in Theorem 2.1 blows up in finite time.*

*Proof.* Using Lemma 3.1, we know that

$$\hat{\rho}(\rho, v) = \frac{1}{c^2}\tilde{\rho}v^2 + \rho,$$

thus,

$$\hat{\rho}(\bar{\rho}, 0) = \bar{\rho}. \tag{3.3}$$

This implies that

$$\rho - \bar{\rho} = (\hat{\rho}(\rho, v) - \hat{\rho}(\bar{\rho}, 0)) - \frac{1}{c^2}\tilde{\rho}v^2. \tag{3.4}$$

From the first equation of (3.2), it is easy to see that

$$\frac{d}{dt}M(t) = \int (\hat{\rho}(\rho, v) - \hat{\rho}(\bar{\rho}, 0))_t \, dx$$

$$= \int \hat{\rho}_t \, dx$$

$$= -\int \nabla_x \bullet (\tilde{\rho}v) \, dx$$

$$= 0,$$

where we have used the first equation in (2.2) or (2.15). Hence,

$$M(t) = M(0) = \int (\hat{\rho}_0(x) - \bar{\rho}) \, dx > 0. \tag{3.5}$$

Using the second equation in (2.2) and (3.2), we compute

$$F'(t) = \int (\tilde{\rho}v)_t \bullet x \, dx$$

$$= -\int [\nabla \bullet (\tilde{\rho}v \otimes v) + \nabla p(\rho)] \bullet x \, dx. \tag{3.6}$$

But if $\bar{p} = p(\bar{\rho})$, we have

$$\nabla p \bullet x = \nabla(p - \bar{p}) \bullet x$$
$$= \nabla \bullet [x(p - \bar{p})] - (p - \bar{p})\nabla \bullet x \tag{3.7}$$
$$= \nabla \bullet [x(p - \bar{p})] - 3(p - \bar{p}).$$

We also note that

$$
\begin{aligned}
\nabla \bullet (\tilde{\rho} v \otimes v) \bullet x &= \sum_{i,j=1}^{3} \partial_{x_i} (\tilde{\rho} v_i v_j) x_j \\
&= \sum_{i,j=1}^{3} [\partial_{x_i} (\tilde{\rho} v_i v_j x_j) - \tilde{\rho} v_i v_j \partial_{x_i} x_j] \\
&= \sum_{i,j=1}^{3} [\partial_{x_i} (\tilde{\rho} v_i v_j x_j) - \tilde{\rho} v_i v_j \delta_{ij}] \\
&= -\tilde{\rho} v^2 + \sum_{j=1}^{3} \nabla \bullet (\tilde{\rho} v \otimes v x_j),
\end{aligned}
\tag{3.8}
$$

where $v^2 = v^T v$. Inserting (3.7) and (3.8) into (3.6), and using the divergence theorem, we obtain

$$
F'(t) = \int \tilde{\rho} v^2 \, dx + 3 \int (p - \bar{p}) \, dx.
\tag{3.9}
$$

Since $p''(\rho) \geq 0$, $p'(\rho)$ is a non-decreasing function of $\rho$. It is clear that

$$
p(\rho) - p(\bar{\rho}) = \int_{\bar{\rho}}^{\rho} p'(\xi) \, d\xi \geq p'(\bar{\rho})(\rho - \bar{\rho}).
\tag{3.10}
$$

Thus, using (3.4) and (3.5), one has

$$
\begin{aligned}
F'(t) &\geq \int \tilde{\rho} v^2 \, dx + 3s^2 \int (\rho - \bar{\rho}) \, dx \\
&= \int \tilde{\rho} v^2 \, dx + 3s^2 M(t) - \frac{3s^2}{c^2} \int \tilde{\rho} v^2 \, dx \\
&= \left(1 - \frac{3s^2}{c^2}\right) \int \tilde{\rho} v^2 \, dx + 3s^2 M(0) \\
&\geq \left(1 - \frac{3s^2}{c^2}\right) \int \tilde{\rho} v^2 \, dx.
\end{aligned}
\tag{3.11}
$$

On the other hand, we have the following estimate:

$$
\begin{aligned}
F^2(t) &= \left(\int \tilde{\rho} v \bullet x \, dx\right)^2 \\
&\leq \left(\int_{B(t)} |x|^2 \tilde{\rho} \, dx\right)\left(\int_{B(t)} \tilde{\rho} v^2 \, dx\right) \\
&\leq 2\left(\int_{B(t)} |x|^2 \hat{\rho} \, dx\right)\left(\int_{B(t)} \tilde{\rho} v^2 \, dx\right),
\end{aligned}
\tag{3.12}
$$

where we have used the following fact:

$$
\tilde{\rho} \leq 2\hat{\rho}.
\tag{3.13}
$$

To see this, we note that from the subluminal condition $p'(\rho) < c^2$, together with $p(0) = 0$, we get $p(\rho) \leq c^2 \rho$. Thus

$$
\tilde{\rho} = \frac{1}{c^2} \tilde{\rho} v^2 + \rho + \frac{p}{c^2} \leq \frac{1}{c^2} \tilde{\rho} v^2 + 2\rho = \hat{\rho} + \rho \leq 2\hat{\rho}.
$$

Due to (3.3) and (3.5), we have the following estimate:

$$\int_{B(t)} \hat{\rho}|x|^2 \, dx \leq (R+st)^2 \int_{B(t)} \hat{\rho} \, dx$$
$$= (R+st)^2(M(t) + \int_{B(t)} \bar{\rho} \, dx)$$
$$= (R+st)^2(M(0) + \int_{B(t)} \bar{\rho} \, dx) \tag{3.14}$$
$$= (R+st)^2 \int_{B(t)} \hat{\rho}_0(x) \, dx$$
$$\leq \frac{4\pi}{3}(R+st)^5(\max \hat{\rho}_0(x)).$$

Hence, (3.12) gives

$$F^2(t) \leq \frac{8\pi}{3}(R+st)^5(\max \hat{\rho}_0(x))(\int_{B(t)} \tilde{\rho}v^2 \, dx)$$
$$\equiv K_0(R+st)^5(\int_{B(t)} \tilde{\rho}v^2 \, dx), \tag{3.15}$$

where

$$K_0 = \frac{8\pi}{3}(\max \hat{\rho}_0(x)). \tag{3.16}$$

Thus (3.11) and (3.15) imply that

$$F'(t) \geq \left(1 - \frac{3s^2}{c^2}\right)K_0^{-1}(R+st)^{-5}F^2(t), \tag{3.17}$$

so

$$\frac{F'}{F^2} \geq K_1(R+st)^{-5}, \tag{3.18}$$

where $K_1 = (1 - \frac{3s^2}{c^2})K_0^{-1}$. Integrating (3.18) with respect to $t$, one has

$$\frac{1}{F(t)} \leq \frac{1}{F(0)} - \frac{K_1}{4s}[R^{-4} - (R+st)^{-4}] \equiv \psi(t). \tag{3.19}$$

Now $\psi(0) = \frac{1}{F(0)} > 0$ by assumption, and

$$\psi(+\infty) = \frac{1}{F(0)} - \frac{K_1}{4s}R^{-4} < 0,$$

if

$$F(0) > \frac{4sR^4}{K_1} \equiv \Gamma. \tag{3.20}$$

Therefore,

$$\frac{1}{F(t_0)} = 0, \quad for \ some \ t_0 > 0. \tag{3.21}$$

Thus the life-span $T$ of smooth solutions satisfies $T < t_0$. This completes the proof of Theorem 3.2. $\quad \square$

## 4. Singularity Formation: Finite Energy Case

Due to the hyperbolic nature of Einstein equations, one expects the finite propagation speed of waves in the solutions. We will prove in the following lemma that for any smooth solution with compactly supported initial data, the support of the solution is invariant in time.

**Lemma 4.1.** *Let $(\rho, v)(x, t)$ be a smooth solution of the Cauchy problem (1.3)–(1.4) up to some time $T > 0$. If the support of initial data is contained in the ball $B_R(0)$ centered at the origin with radius $R$, then the support of $(\rho, v)(x, t)$ is contained in the same ball $B_R(0)$ for any $t \in [0, T)$.*

*Proof.* Assume that the initial support of the solution is contained in a ball $B_R(0)$, the support of the smooth solution will remain compact by the hyperbolic nature of the system (1.3). We denote by $x(t; x_0)$ the particle path starting at $x_0$ when $t = 0$, i.e.,

$$\frac{d}{dt} x(t; x_0) = v(x(t; x_0), t), \quad x(t = 0; x_0) = x_0, \tag{4.1}$$

and by $S_p(t)$ the closed region that is the image of $B_R(0)$ under the flow map (4.1). Hence, the support of the smooth solution of (1.3)–(1.4) will remain inside $S_p(t)$. Thus, fixing any $x_0$ on the boundary of $B_R(0)$, we have $\rho_0(x_0) = 0$ and $v_0(x_0) = 0$, and $x(t; x_0)$ is on the boundary of $S_p(t)$. Furthermore,

$$\frac{d}{dt} x(t; x_0) = v(x(t; x_0), t) = 0, \tag{4.2}$$

due to continuity of $v(x, t)$ and the fact that $x(t; x_0)$ sits at the boundary of the support of the solution. Therefore, $x(t; x_0) = x_0$ for any $t \in [0, T)$ whenever $|x_0| = R$. Hence, $S_p(t) = B_R(0)$. This proves this lemma.  ☐

Based on Lemma 4.1, we shall prove the following blowup result.

**Theorem 4.2.** *Suppose the support of the smooth functions $(\rho_0(x), v_0(x))$ is non-empty and contained in a ball $B_R(0)$ centered at the origin with radius $R$. Then the smooth solution of (1.3)-(1.4) with the initial data $(\rho_0(x), v_0(x))$ blows up in finite time.*

*Proof.* We first introduce the following functions:

$$H(t) = \frac{1}{2} \int \hat{\rho}|x|^2 \, dx, \quad F(t) = \int \tilde{\rho} v \bullet x \, dx, \quad E(t) = \int \hat{\rho} \, dx. \tag{4.3}$$

Here, $H(t)$ is the second moment of $\hat{\rho}$, $F(t)$ is the total radial "generalized" momentum, and $E(t)$ is the total "generalized" energy. These functions are well defined in the domain where the smooth solutions exist. Interesting relations between them can be obtained by the following calculations.

$E(t)$ is conserved, because using (2.2) one has

$$E'(t) = \int \hat{\rho}_t \, dx = - \int \nabla_x \bullet (\tilde{\rho} v) \, dx = 0.$$

We thus have

$$E(t) = E(0) = \int \hat{\rho}_0(x) \, dx > 0, \tag{4.4}$$

for non-trivial initial data.

For $H(t)$, we have

$$
\begin{aligned}
H'(t) &= \frac{1}{2} \int \hat{\rho}_t |x|^2 \, dx \\
&= -\frac{1}{2} \int [\nabla_x \bullet (\tilde{\rho} v)] |x|^2 \, dx \\
&= \int \tilde{\rho} v \bullet x \, dx \\
&= F(t),
\end{aligned}
\tag{4.5}
$$

where we have used the relation

$$
[\nabla_x \bullet (\tilde{\rho} v)] |x|^2 = \nabla_x \bullet (\tilde{\rho} v |x|^2) - \tilde{\rho} v \bullet 2x.
$$

From the second equation in (2.2) and integrating by parts, we have

$$
\begin{aligned}
H''(t) = F'(t) &= \int (\tilde{\rho} v)_t \bullet x \, dx \\
&= -\int ([\nabla_x \bullet (\tilde{\rho} v \otimes v)] \bullet x + (\nabla_x p \bullet x)) \, dx \\
&= \int \tilde{\rho} v^2 \, dx + \int 3p \, dx \\
&= c^2 \int_{B_R(0)} (\frac{1}{c^2} \tilde{\rho} v^2 + \frac{3p}{c^2}) \, dx.
\end{aligned}
\tag{4.6}
$$

By Jensen's inequality, we have

$$
\begin{aligned}
\int_{B_R(0)} p(\rho) \, dx &= \left( \frac{4\pi}{3} R^3 \right) \frac{\int_{B_R(0)} p(\rho) \, dx}{(\frac{4\pi}{3} R^3)} \\
&\geq (\frac{4\pi}{3} R^3) p \left( \frac{\int_{B_R(0)} \rho \, dx}{(\frac{4\pi}{3} R^3)} \right),
\end{aligned}
\tag{4.7}
$$

so (4.6) and (4.7) imply

$$
\begin{aligned}
H''(t) &\geq c^2 \left[ \int_{B_R(0)} \frac{1}{c^2} \tilde{\rho} v^2 dx + \frac{3}{c^2} (\frac{4\pi}{3} R^3) p(\rho_B) \right] \\
&\equiv c^2 N(t),
\end{aligned}
\tag{4.8}
$$

where

$$
\rho_B = \frac{\int_{B_R(0)} \rho \, dx}{(\frac{4\pi}{3} R^3)}
$$

is the mean density over $B_R(0)$. Since

$$
E(t) = E(0) = \int_{B_R(0)} \left( \frac{1}{c^2} \tilde{\rho} v^2 + \rho \right) \, dx,
$$

it is possible to bound $N(t)$ from below using $E(0)$. We consider two cases. First if

$$\int_{B_R(0)} \frac{1}{c^2} \tilde{\rho} v^2 \, dx \geq \frac{1}{2} E(0),$$

we get

$$N(t) \geq \frac{1}{2} E(0). \tag{4.9}$$

On the other hand, if

$$\int_{B_R(0)} \frac{1}{c^2} \tilde{\rho} v^2 \, dx \leq \frac{1}{2} E(0),$$

then as

$$E(t) = E(0) = \int_{B_R(0)} \left( \frac{1}{c^2} \tilde{\rho} v^2 + \rho \right) \, dx \leq \frac{1}{2} E(0) + \int_{B_R(0)} \rho \, dx,$$

we have

$$\int_{B_R(0)} \rho \, dx \geq \frac{1}{2} E(0).$$

Thus

$$\begin{aligned}
N(t) &\geq \frac{3}{c^2} (\frac{4\pi}{3} R^3) p(\rho_B) \\
&\geq \frac{3}{c^2} (\frac{4\pi}{3} R^3) p(\frac{1}{2} E_B(0)) \\
&\equiv B_1 E(0) > 0,
\end{aligned} \tag{4.10}$$

where $B_1 = \frac{4\pi R^3}{c^2 E(0)} p(\frac{1}{2} E_B(0))$, and $E_B(0) = \frac{E(0)}{(\frac{4\pi}{3} R^3)}$. Define $B = c^2 \min\{\frac{1}{2}, B_1\}$; then (4.8)–(4.10) imply that

$$H''(t) \geq B E(0) > 0. \tag{4.11}$$

This gives a lower bound on $H(t)$:

$$H(t) \geq \frac{1}{2} B E(0) t^2 + F(0) t + H(0). \tag{4.12}$$

In order to refine (4.12), we estimate $F(t)$ in terms of $H(t)$ and $E(t)$. Using (3.10), we have

$$\begin{aligned}
|F(t)| &= |\int (\tilde{\rho} v \bullet x) \, dx| \\
&\leq (\int \tilde{\rho} |x|^2 \, dx)^{\frac{1}{2}} (\int \tilde{\rho} v^2 \, dx)^{\frac{1}{2}} \\
&\leq \sqrt{2} H(t)^{\frac{1}{2}} (c^2 E(t))^{\frac{1}{2}} \\
&= c\sqrt{2} [H(t) E(t)]^{\frac{1}{2}} \\
&\equiv D[H(t) E(t)]^{\frac{1}{2}}.
\end{aligned} \tag{4.13}$$

We derive from (4.12) and (4.13) that

$$H(t) \geq \frac{1}{2} BE(0)t^2 - D[H(0)E(0)]^{\frac{1}{2}}t + H(0). \tag{4.14}$$

We note that (4.14) implies that $H(t)$ tends to infinity as $t$ goes to infinity. However, we have the following uniform upper bound for $H(t)$:

$$
\begin{aligned}
H(t) &= \frac{1}{2} \int \hat{\rho}|x|^2 \, dx \\
&= \frac{1}{2} \int_{B_R(0)} \hat{\rho}|x|^2 \, dx \\
&\leq \frac{1}{2} R^2 \int \hat{\rho} \, dx \\
&= \frac{1}{2} E(0) R^2.
\end{aligned} \tag{4.15}
$$

Thus (4.12) or (4.14) together with (4.15) imply that the life-span of the smooth solutions must be finite if $E(0) > 0$. This completes the proof of Theorem 4.2. $\square$

Notice that, for non-trivial initial data, we have

$$
\begin{aligned}
&H(0) - \frac{1}{2} E(0) R^2 \\
&= \frac{1}{2} \int \hat{\rho}_0 |x|^2 \, dx - \frac{1}{2} R^2 \int \hat{\rho}_0 \, dx \\
&= \frac{1}{2} \int_{B_R(0)} \hat{\rho}_0 (|x|^2 - R^2) \, dx \\
&< 0.
\end{aligned} \tag{4.16}
$$

This enables us to estimate the life-span as follows: from (4.14) and (4.15) we have for smooth solutions

$$\frac{1}{2} BE(0)t^2 - D\sqrt{H(0)E(0)}t + H(0) \leq \frac{1}{2} E(0) R^2.$$

This is equivalent to

$$\phi(t) = Bt^2 - 2Ddt + 2d^2 - R^2 \leq 0, \tag{4.17}$$

where $d^2 = \frac{H(0)}{E(0)}$, and $\phi(0) < 0$ by (4.16). Hence, the life-span $T$ of the smooth solution satisfies

$$T \leq \frac{Dd + \sqrt{D^2 d^2 - 2Bd^2 + R^2 B}}{B}. \tag{4.18}$$

## 5. Concluding Remarks

We have proved the blowup of smooth solutions of relativistic Euler equations in both cases: finite initial energy (Theorem 4.2) and infinite initial energy (Theorem 3.2). In contrast to the characteristic method, we adapted the approach via some functions: total "generalized" energy, total radial "generalized" momentum, and the second moment. Our approach depends on the beautiful structure of the equations and several quantities constructed from the natural variables; cf. Lemma 3.1. Although the relativistic Euler equations are much more complicated than the classical Euler equations, these structures make our proofs possible. We will now make some remarks on our results and discuss some related issues.

*Remark 1.* In our blowup theorems, the velocity in the far field is assumed to be zero initially. For the more general case, say $v_0(x) = \bar{v}$ off a bounded set, the change of variables (Sideris [11])

$$v \to v - \bar{v}, \ x \to x + t\bar{v}$$

will reduce this problem to the case we considered.

*Remark 2.* The condition

$$p'(\bar{\rho}) < \frac{c^2}{3}$$

in Theorem 3.2 arises naturally in the proof. Here, 3 is the spatial dimension. In $d$ dimensions, 3 is replaced by $d$. In particular, for $d = 1$, this condition is that the sound speed is subluminal. For $p(\rho) = \sigma^2 \rho$, $d = 1$, this condition guarantees the genuinely nonlinearity of the relativistic Euler equations and allows the existence of global solutions in BV; see Smoller and Temple [14]. This condition is not required in Theorem 4.2.

*Remark 3.* Our blowup results crucially depend on the compact support of the perturbations. Singularity formation for more general initial data remains open.

*Remark 4.* The type of singularity which occurs is another open problem. The possibilities are: a) shock formation, b) violation of the subluminal conditions; e.g. $|v|$ tends to $c$, or $p'(\rho) \to c^2$, c) concentration of the mass.
    For $p(\rho) = \sigma^2 \rho$ and $d = 1$, the singularity must be a shock if the initial data is away from the vacuum. It was shown in Smoller and Temple [14] that weak solutions exist globally in time with bounded total variation, subluminal velocity and positive density uniformly bounded from above and below. Furthermore, Smoller and Temple proved in [14] that the subluminal condition guarantees the genuine nonlinearity of the equations, so one concludes that the singularities in the solutions must be shocks by Lax' theory [4]. It would be interesting to clarify the types of singularities for relativistic Euler equations in multi-dimensions. However, black hole formation is impossible for our problem, since our spacetime is fixed to be flat Minkowski spacetime.

*Remark 5.* The singularity in our Theorem 3.2 looks like shock formation. The largeness condition in radial "generalized" momentum, (3.20), implies that the particle velocity must be supersonic in some region relative to the sound speed at infinity. One can guess that the singularity formation is detected as the disturbance overtaking the wave front

thereby forcing the front to propagate with supersonic speed. To see these things, we argue as follows. Using the fact $\tilde{\rho} \le 2\hat{\rho}$(cf. (3.13)), one has

$$F(0) \le \frac{8\pi}{3} R^4 (\max \hat{\rho}_0(x)) \max |v_0(x)|, \tag{5.1}$$

while

$$\begin{aligned} \Gamma &= \frac{4s}{1 - \frac{3s^2}{c^2}} \frac{8\pi}{3} R^4 \max \hat{\rho}_0(x) \\ &\ge 4s \frac{8\pi}{3} R^4 (\max \hat{\rho}_0(x)). \end{aligned} \tag{5.2}$$

Hence, $F(0) > \Gamma$ implies that

$$\max |v_0(x)| \ge 4s. \tag{5.3}$$

This insures the initial particle velocity is supersonic in some region. However, the rigorous proof of shock formation is still open.

*Remark 6.* The lower bound of the initial radial "generalized" momentum in (3.20) depends on the initial velocity through $\hat{\rho}$. This is different from the Newtonian case, where $\hat{\rho}$ is replaced by $\rho$ and so in the Newtonian case it does not depend on the velocity. On the other hand, the velocity has to be subluminal. Therefore, we must show that the set of initial data required in Theorem 3.2 is non-empty. From (5.1)–(5.3), we find the set is non-empty if

$$c > \max |v_0(x)| \ge \frac{4s}{1 - \frac{3s^2}{c^2}}. \tag{5.4}$$

A simple calculation shows that the necessary condition for $s$ to satisfy is

$$s < \left( \frac{\sqrt{7}}{3} - \frac{2}{3} \right) c. \tag{5.5}$$

Since $\tilde{\rho} \ge \hat{\rho}$, $F(0)$ is of the same order as the upper bound in (5.1). Thus, if $s$ is chosen to be small (this can be done by choosing $\bar{\rho}$ small), one can easily find initial data satisfying the conditions required in Theorem 3.2.

*Remark 7.* The equation of state $p(\rho)$ satisfying (1.5) is quite general for isentropic fluids. It can be weakened by replacing $p''(\rho) \ge 0$ with $p'(\rho)$ non-decreasing. This includes the well-known $\gamma$-law, $p(\rho) = \sigma^2 \rho^\gamma$, $\gamma \ge 1$ as a particular case. In fact, in the case of a $\gamma$-law, (3.13) can be refined, and thus (3.20) can be replaced by a weaker condition, as we now show.

When $\gamma = 1$, $s = \sigma$. Equation (3.13) is refined as $\tilde{\rho} < (1 + \frac{\sigma^2}{c^2})\hat{\rho} = \tilde{\rho} + \frac{\sigma^2}{c^4}\tilde{\rho}v^2$ by Lemma 3.1. Thus, (3.20) can be weakened to:

$$F(0) > \Gamma_1 = \frac{4\sigma}{1 - \frac{3\sigma^2}{c^2}} \left( 1 + \frac{\sigma^2}{c^2} \right) \frac{4\pi}{3} R^4 \max \hat{\rho}_0(x). \tag{5.6}$$

When $\gamma > 1$, we observe that the subluminal condition

$$p'(\rho) = \gamma \sigma^2 \rho^{\gamma - 1} \le c^2$$

implies that $\frac{p}{c^2} \leq \frac{1}{\gamma}\rho$. Thus,

$$
\begin{aligned}
\left(1 + \frac{1}{\gamma}\right)\hat{\rho} = \hat{\rho} &+ \frac{1}{\gamma}\rho + \frac{1}{\gamma c^2}\tilde{\rho}v^2 \\
&\geq \hat{\rho} + \frac{1}{\gamma}\rho \\
&\geq \hat{\rho} + \frac{p}{c^2} = \tilde{\rho}.
\end{aligned}
$$

We can thus refine (3.13) to $\tilde{\rho} < (1 + \frac{1}{\gamma})\hat{\rho}$, and then (3.20) is replaced by the following weaker condition:

$$
F(0) > \Gamma_2 = \frac{4s}{1 - \frac{3s^2}{c^2}}\left(1 + \frac{1}{\gamma}\right)\frac{4\pi}{3}R^4 \max \hat{\rho}_0(x). \tag{5.7}
$$

## 6. Appendix

For the reader's convenience, we justify the construction of a strictly convex entropy function for (1.3) due to Makino and Ukai in [7], and we will also correct several errors. To this end, we first record (2.6)–(2.7) here ,

$$
\theta_t + \sum_{k=1}^{3}(f^k(\theta))_{x_k} = 0, \tag{6.1}
$$

where $\theta = (\theta_0, \theta_1, \theta_2, \theta_3)^T$ and $f^k(\theta) = (\theta_k, f_1^k, f_2^k, f_3^k)$ are defined by

$$
\begin{aligned}
\theta_0 &= \hat{\rho}, \ \theta_j = \tilde{\rho}v_j, \\
f_j^k &= \tilde{\rho}v_j v_k + p\delta_{jk}, \ j = 1, 2, 3.
\end{aligned} \tag{6.2}
$$

The scalar function $\eta = \eta(\theta)$ is called an entropy function and scalar functions $q^k(\theta)$, $k = 1, 2, 3$ are called entropy flux functions, if they satisfy:

$$
\nabla_\theta \eta(\theta)\nabla_\theta f^k(\theta) = \nabla_\theta q^k(\theta). \tag{6.3}
$$

Since the the right-hand side of (6.3) is a gradient of the function $q^k$, the relevant integrability condition (cf. [1], p. 39) is

$$
(\nabla_\theta^2 \eta)(\nabla_\theta f^k) = (\nabla_\theta f^k)^T(\nabla_\theta^2 \eta). \tag{6.4}
$$

If we find such an $\eta$ that is strictly convex, the change of variables $\theta \to w = (\nabla_\theta \eta)^T$ will render (6.1) into the symmetric form (2.17); see [1], where

$$
A^0 = (\nabla_\theta^2 \eta)^{-1}, \ A^k = (\nabla_\theta f^k)(\nabla_\theta^2 \eta)^{-1}. \tag{6.5}
$$

To see this, we apply chain rule:

$$
\partial_\alpha \theta = (\nabla_\theta w)^{-1}\partial_\alpha w = (\nabla_\theta^2 \eta)^{-1}\partial_\alpha w. \tag{6.6}
$$

Substituting (6.6) into (6.1), we obtain

$$(\nabla_\theta^2 \eta)^{-1} w_t + \sum_{k=1}^{3} (\nabla_\theta f^k)(\nabla_\theta^2 \eta)^{-1} w_{x_k} = 0.$$

$A^0$ is positive definite if and only if $\eta$ is strictly convex. To verify the real symmetry of $A^k$, we use (6.4). Multiplying both sides of (6.4) by $(\nabla_\theta^2 \eta)^{-1}$ on the left and right, we see $A^k = (A^k)^T$.

We will solve (6.3) keeping the mechanical energy of classical Euler equations in mind. Thus, instead of $\theta$, we will use $z = (\rho, v_1, v_2, v_3)^T$ as independent variables. We compute:

$$\nabla_z \theta = \begin{pmatrix} B_1 & B_2 v^T \\ B_3 v & B_2 v v^T + B_4 I_3 \end{pmatrix}, \tag{6.7}$$

where

$$B_1 = \frac{c^2 + p'}{c^2 - v^2} - \frac{p'}{c^2}, \quad B_2 = 2\frac{\rho c^2 + p}{(c^2 - v^2)^2}, \tag{6.8}$$
$$B_3 = \frac{c^2 + p'}{c^2 - v^2}, \quad B_4 = \frac{\rho c^2 + p}{c^2 - v^2}.$$

Moreover,

$$det(\nabla_z \theta) = \frac{(\rho c^2 + p)^3 (c^4 - v^2 p')}{c^2 (c^2 - v^2)^4} > 0, \tag{6.9}$$

in $\Omega_z$. We can thus compute the inverse of $\nabla_z \theta$:

$$(\nabla_z \theta)^{-1} = \begin{pmatrix} c^2 (c^2 + v^2) E_1 & -2c^2 E_1 v^T \\ -c^2 (c^2 + p') E_1 E_2 v & 2p' E_1 E_2 v v^T + E_2 I_3 \end{pmatrix}, \tag{6.10}$$

with

$$E_1 = \frac{1}{c^4 - p' v^2}, \quad E_2 = \frac{c^2 - v^2}{\rho c^2 + p}. \tag{6.11}$$

Based on (6.10), we will solve (6.3) using $z$ as independent variables for convenience. In the $z$-variables, (6.3) becomes

$$\nabla_z \eta C^k = D_z q^k, \ k = 1, 2, 3, \tag{6.12}$$

where

$$(\nabla_z f^k) = \begin{pmatrix} B_3 v_k & B_2 v_k v^T + B_4 e_k^T \\ B_3 v_k v + p' e_k & B_2 v_k v v^T + B_4 v e_k^T + B_4 v_k I_3 \end{pmatrix}, \tag{6.13}$$

and

$$C^k = (\nabla_z \theta)^{-1} (\nabla_z f^k)$$
$$= \begin{pmatrix} c^2 C_1 v_k & C_3 e_k^T \\ -C_1 C_2 v_k v + C_2 e_k & -C_4 v e_k^T + v_k I_3 \end{pmatrix}, \tag{6.14}$$

with

$$C_1 = \frac{c^2 - p'}{c^4 - p'v^2}, \quad C_2 = p'E_2 = \frac{p'(c^2 - v^2)}{\rho c^2 + p},$$
$$C_3 = \frac{c^2(\rho c^2 + p)}{c^4 - v^2 p'}, \quad C_4 = \frac{p'(c^2 - v^2)}{c^4 - p'v^2}. \tag{6.15}$$

Formally, (6.12) is an over-determined system, consisting of 12 equations for 4 unknowns. We seek solutions with the special form:

$$\eta = \eta(\rho, y), \quad q^k = Q(\rho, y)v_k, \quad y = v^2 = v_1^2 + v_2^2 + v_3^2 \tag{6.16}$$

to reduce the number of equations in (6.12). Substituting this ansatz into (6.12), we obtain the following first order linear system:

$$\begin{cases} \eta_y = Q_y, \\ c^2 C_1 \eta_\rho + 2C_2(1 - C_1 y)\eta_y = Q_\rho, \\ C_3 \eta_\rho - 2C_4 y \eta_y = Q. \end{cases} \tag{6.17}$$

This seems still an over-determined system. However, it is possible to derive a decoupled equation for $Q$ from (6.17). We first multiply the second equation of (6.17) by $(\rho^2 c^2 + p)$, and using $(c^2 - p')C_3 = c^2 C_1(\rho^2 c^2 + p)$, we have

$$(c^2 - p')C_3 \eta_\rho + 2C_2(\rho c^2 + p)(1 - C_1 y)\eta_y = (\rho c^2 + p)Q_\rho. \tag{6.18}$$

Then, we compute $(c^2 - p') \times (6.17)_3$:

$$(c^2 - p')C_3 \eta_\rho - 2(c^2 - p')C_4 y \eta_y = (c^2 - p')Q. \tag{6.19}$$

We subtract (6.19) from (6.18) and substitute $\eta_y$ with $Q_y$, using (6.15); this reduces (6.17) into the following decoupled system:

$$\begin{cases} \eta_y = Q_y, \\ 2(c^2 - y)p'Q_y = (\rho c^2 + p)Q_\rho - (c^2 - p')Q. \end{cases} \tag{6.20}$$

We now proceed to solve (6.20) with the help of (6.17). First, $(6.20)_1$ gives

$$\eta = Q(\rho, y) + G(\rho). \tag{6.21}$$

Substitute this into $(6.17)_3$ to get

$$G_\rho = \frac{c^2 - y^2}{\rho c^2 + p}Q - \frac{c^2 - y^2}{c^2}Q_\rho,$$

or equivalently

$$G_\rho = \frac{1}{\rho c^2 + p}q - \frac{1}{c^2}q_\rho, \quad q = (c^2 - y)Q. \tag{6.22}$$

We observe that $G$ depends on $\rho$ only, so we have a linear first order ODE for $q$,

$$\frac{1}{\rho c^2 + p}q - \frac{1}{c^2}q_\rho = \frac{f(\rho)}{c^2},$$

which has the solution

$$q(\rho) = e^{\phi(\rho)}(g(\rho) + h(y)). \tag{6.23}$$

Here $\phi(\rho)$ is defined in (2.8). Substituting (6.23) into (6.20)$_2$, and separating variables, one has

$$\frac{\rho c^2 + p}{p'}\frac{dg}{d\rho} - g = 2(c^2 - y)\frac{dh}{dy} + h = m(\rho, y), \tag{6.24}$$

where the first term in (6.24) is independent of $y$, while the second term is independent of $\rho$. Thus, $m$ is independent of both $\rho$ and $y$. We conclude that $m = const$. Thus, by integrating (6.24), we have

$$\begin{aligned}
q &= D_1(\rho c^2 + p) + D_2 e^{\phi(\rho)}\sqrt{c^2 - v^2}, \\
G &= -\frac{D_1}{c^2}p + D_3, \\
Q &= D_1\frac{\rho c^2 + p}{c^2 - v^2} + D_2\frac{e^{\phi(\rho)}}{\sqrt{c^2 - v^2}}, \\
\eta &= D_1\frac{\rho c^2 + p}{c^2 - v^2} + D_2\frac{e^{\phi(\rho)}}{\sqrt{c^2 - v^2}} - \frac{D_1}{c^2}p + D_3,
\end{aligned} \tag{6.25}$$

where $D_1$, $D_2$ and $D_3$ are integration constants. With $K$ as in (2.8), one choice is

$$D_1 = c^2, \ D_2 = -cK, \ D_3 = 0,$$

thus

$$\eta = c^2\hat{\rho} - \frac{cKe^{\phi(\rho)}}{\sqrt{c^2 - v^2}}. \tag{6.26}$$

The associated entropy-flux is $(q^1, q^2, q^3)^T$ defined by

$$q^k = \frac{c^2(\rho c^2 + p)}{c^2 - v^2}v_k - \frac{cKe^{\phi(\rho)}}{\sqrt{c^2 - v^2}}v_k. \tag{6.27}$$

Moreover $\eta$ is strictly convex as was verified in Sect. 2.

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