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Zero dissipation limit to a Riemann solution consisting of two shock waves for the 1D compressible isentropic Navier-Stokes equations

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Abstract We investigate the zero dissipation limit problem of the one dimensional compressible isentropic Navier-Stokes equations with Riemann initial data in the case of the composite wave of two shock waves. It is shown that the unique solution to the Navier-Stokes equations exists for all time, and converges to the Riemann solution to the corresponding Euler equations with the same Riemann initial data uniformly on the set away from the shocks, as the viscosity vanishes. In contrast to previous related works, where either the composite wave is absent or the effects of initial layers are ignored, this gives the first mathematical justification of this limit for the compressible isentropic Navier-Stokes equations in the presence of both composite wave and initial layers. Our method of proof consists of a scaling argument, the construction of the approximate solution and delicate energy estimates.

Keywords zero dissipation limit, compressible Navier-Stokes equations, shock waves, initial layers

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1 Introduction

The asymptotic behavior of viscous flows in vanishing dissipation limit process is one of the important, longstanding problems in the theory of compressible fluid flow. It is expected that the solution to viscous flows should converge strongly, when dissipation vanishes, to the solution to the corresponding inviscid flow. When the solution to the inviscid flow is smooth, this problem can be solved by classical Hilbert expansion along with energy method. However, the inviscid compressible flow usually contains discontinuities, such as shock waves and contact discontinuities, which have so far prevented solving the problem in the general setting by means of known analytic tools. Essential new ideas and methods are needed to tackle this open problem. Therefore, any attempt on this problem that involves the singularity in the inviscid solution can be viewed as progress to this general program.

In one space dimension, interesting progress has been made on system of hyperbolic conservation laws with artificial viscosity $u_t + f(u)_x = \epsilon u_{xx}$. Using a matched asymptotic expansion method, Goodman

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and Xin [8] proved that, given any piecewise smooth entropy solution with finitely many non-interacting shock waves of the inviscid conservation laws, the above viscous problem admits a sequence of smooth solutions converging to the given invisid solutions in vanishing viscosity limit. Later, Yu [44] improved the results of [8] to allow initial layers by a detailed pointwise analysis. Recently, Zeng [46] proved the large time asymptotic nonlinear stability of a superposition of shock waves with contact waves for the fixed viscosity $\epsilon = 1$. In the context of small BV initial data, the seminal result of Bianchini and Bressan [1] proved the the vanishing viscosity limit of solutions of this viscous hyperbolic system by deriving the uniform BV estimates of solutions independent of the viscosity. This fully settled the problem in small BV case when viscosity matrix is ϵI. However, the problem is still unsolved for physical systems such as the Navier-Stokes equations.

In this paper, we study the zero dissipation limit of the solutions to the Navier-Stokes equations of compressible, isentropic gases which, in Lagrangian formulation, can be written as (see [5]):

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \epsilon \left(\dfrac{u_x}{v} \right)_x, \end{cases} \tag{1.1}$$

where u, v and p denote the fluid velocity, the specific volume, and the pressure in a compressible fluid, respectively, while $\epsilon > 0$ is the constant viscosity coefficient. The pressure p is assumed to be a smooth function of $v > 0$ satisfying

$$p'(v) < 0 < p''(v), \qquad \text{for } v > 0. \tag{1.2}$$

For example, (1.2) holds for polytropic and ideal isothermal gases, for which $p(v) = Cv^{-\gamma}, \gamma \geqslant 1$.

We consider the Cauchy problem for (1.1) with Riemann initial data

$$U(x,0) = \begin{bmatrix} v(x,0) \\ u(x,0) \end{bmatrix} = \begin{cases} U_-, & x < 0, \\ U_+, & x > 0, \end{cases} \tag{1.3}$$

where $U_\pm$ are given constant states. We are especially interested in the relation between the Navier-Stokes solutions, $U^\epsilon(x,t)$ to (1.1) and (1.3), and solutions $U^0(x,t)$ to the corresponding Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0 \end{cases} \tag{1.4}$$

with the same Riemann initial data (1.3).

Although the zero dissipation limit for compressible Naiver-Stokes equations remains as an important open problem, many interesting results were achieved in the past. These results roughly fall into three categories. The first is to use theory of compensated compactness to establish the compactness of the sequence of solutions of the Navier-Stokes equations, and then to extract a subsequence to converge to a limit, which was later justified as a weak solution to the corresponding compressible Euler equations. The representative results are obtained by DiPerna [7] with initial data in H^2, and by Chen and Pereperitsa [4] with suitable smooth initial data. Although the class of initial data in these theories are fairly broad, Riemann data, which are building blocks for the inviscid Euler equations (1.3), are specifically excluded, and the abstract analysis yields little information on the qualitative nature of the viscous solutions. The second kind of results utilizes recent development of nonlinear stability analysis results on elementary waves for compressible Navier-Stokes equations. Motivated by early work of Xin [41] for rarefaction waves, and Goodman and Xin [8] for solution with shock waves, exciting advancement has been made in this direction. It was shown that, given a solution to the compressible Euler equations (1.4) which is piecewise smooth and contains simple wave patterns, there exists a sequence of solutions of the compressible Navier-Stokes equations that converges to the pre-fixed Euler solution in zero dissipation limit. The advantage of this approach is that it can be generalized to general system, and the explicit construction of the viscous solutions gives detailed structure of solutions along with explicit convergence rate. The possible disadvantage of this approach is that this kind of results are often valid only for finite time when shock presents, and the convergence is good only for the preferred (or constructed) sequence of viscous solutions. We refer the readers to [2, 3, 6, 16, 18, 19, 21, 22] for a partial list of results in this direction. Last but not least, Hoff and Liu [9] proposed a framework to study directly the compressible Navier-Stokes equations with Riemann data, established sharp and uniform estimates, analyzed the detailed behavior of the solutions in initial, intermediate, and large time regimes, and finally proved the zero dissipation limit to the Riemann solutions of compressible Euler equation. Comparing with the second category, this program is different in at least four aspects. First, rather than the preferred sequence with approximate initial data, this program shows uniform convergence of Navier-Stokes system with fixed same data as Euler. Second, the stability analysis component in this program has large initial perturbation. Third, this program takes care of both shock waves and initial layers. Finally, the convergence result of this program is globally, not only for a finite time. So far, except for Hoff and Liu [9] where the isentropic Naver-Stokes (1.1) with a single shock wave initial data was solved, no much development appeared in the past two decades. The main motivation of this paper is to extend this result to the case of the composite wave of two shock waves. In particular, we prove that the solutions of the compressible isentropic Navier-Stokes system (1.1) with Riemann initial data (1.3) exist for all time, and converge to the Riemann solution to the Euler equations with the same Riemann initial data that is a composite wave of two shock waves, as the viscosity tends to zero. This gives the first mathematical justification of this limit for the compressible isentropic Navier-Stokes equations in the presence of both composite wave and initial layers. For other related works, the readers are referred to [23–25, 30, 31, 42, 43, 48].

Now, we introduce some preliminary notations and give some background materials before stating the main theorem. It is known that the system (1.4) has two eigenvalues: $\lambda_1 = -\sqrt{-p'(v)} < 0$, $\lambda_2 = \sqrt{-p'(v)} > 0$, where two characteristic fields are genuinely nonlinear. In the present paper, we focus our attention on the situation, where the Riemann solution to (1.4) and (1.3) is a composite wave of two shock waves (and three constants states) (see Figure 1):

$$U^0(x,t) = \begin{cases} U_-, & x < s_1 t, \\ U_m, & s_1 t < x < s_2 t, \\ U_+, & x > s_2 t. \end{cases} \tag{1.5}$$

Here, U_m is the intermediate state and the shock speeds s_1 and s_2 are constants determined by the Rankine-Hugoniot condition

$$\begin{aligned} &s_1(v_m - v_-) = u_- - u_m, \quad s_1(u_m - u_-) = p(v_m) - p(v_-), \\ &s_2(v_+ - v_m) = u_m - u_+, \quad s_2(u_+ - u_m) = p(v_+) - p(v_m) \end{aligned} \tag{1.6}$$

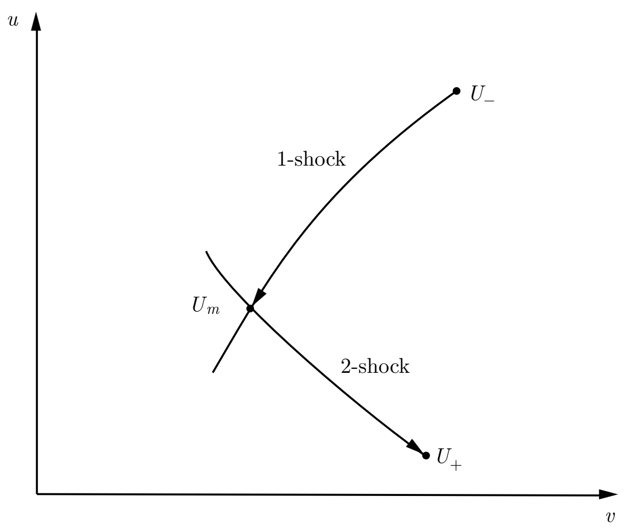

Figure 1 The composite wave of two shock waves

and satisfy entropy conditions

$$-\sqrt{-p'(v_-)} > s_1 > -\sqrt{-p'(v_m)}, \quad \sqrt{-p'(v_m)} > s_2 > \sqrt{-p'(v_+)}. \tag{1.7}$$

To describe the strengths of the shock waves for later use, we set

$$\delta = |U_+ - U_-|, \quad \delta_1 = |U_m - U_-|, \quad \delta_2 = |U_+ - U_m|, \quad \bar{\delta} = \min\{\delta_1, \delta_2\}. \tag{1.8}$$

When we choose δ small in our situation for the fixed U_-, we note that it holds

$$\delta \leqslant \delta_1 + \delta_2 \leqslant C\delta, \tag{1.9}$$

where C is a positive constant depending only on U_-. Then, if it holds

$$\delta_1 + \delta_2 \leqslant C\bar{\delta}, \quad \text{as} \quad \delta_1 + \delta_2 \to 0, \tag{1.10}$$

for some positive constant C, we call the strengths of the shock waves "small with same order". In what follows, we always assume (1.10).

Next, we recall the definitions of viscous shock waves of (1.1), which correspond to the above shock waves. We see that the 1-viscous shock wave which corresponds to the 1-shock wave is a traveling wave solution to (1.1) with the formula $\bar{U}_1^\epsilon(x - s_1t) = (V_1^\epsilon, U_1^\epsilon)^t(x - s_1t)$, which is determined by

$$\begin{cases} -s_1(V_1^\epsilon)' - (U_1^\epsilon)' = 0, \\ -s_1(U_1^\epsilon)' + p(V_1^\epsilon)' = \epsilon\left[\dfrac{(U_1^\epsilon)'}{V_1^\epsilon}\right]', \\ (V_1^\epsilon, U_1^\epsilon)^t(-\infty) = U_-, \\ (V_1^\epsilon, U_1^\epsilon)^t(+\infty) = U_m, \end{cases} \tag{1.11}$$

where $' = \frac{d}{d\xi}$, $\xi = x - s_1t$.

Similarly, the 2-viscous shock wave $\bar{U}_2^\epsilon(x - s_2t) = (V_2^\epsilon, U_2^\epsilon)^t(x - s_2t)$ is defined by

$$\begin{cases} -s_2(V_2^\epsilon)' - (U_2^\epsilon)' = 0, \\ -s_2(U_2^\epsilon)' + p(V_2^\epsilon)' = \epsilon\left[\dfrac{(U_2^\epsilon)'}{V_2^\epsilon}\right]', \\ (V_2^\epsilon, U_2^\epsilon)^t(-\infty) = U_m, \\ (V_2^\epsilon, U_2^\epsilon)^t(+\infty) = U_+, \end{cases} \tag{1.12}$$

where $' = \frac{d}{d\eta}$, $\eta = x - s_2t$.

We denote a composite wave consisting of the two viscous shock waves $(V_i^\epsilon, U_i^\epsilon)^t, i = 1, 2$ by

$$\bar{U}^\epsilon = \begin{bmatrix} \bar{v}^\epsilon \\ \bar{u}^\epsilon \end{bmatrix} = \begin{bmatrix} V_1^\epsilon(x - s_1t) + V_2^\epsilon(x - s_2t) - v_m \\ U_1^\epsilon(x - s_1t) + U_2^\epsilon(x - s_2t) - u_m \end{bmatrix}. \tag{1.13}$$

Since the present paper is concerned with the non-smooth initial perturbation, the integral $\int_{-\infty}^{\infty}(U(x,0) - \bar{U}^1(x,0))dx$ is in general not zero. Fortunately, for weak waves, $0 < \delta_i \ll 1$ $(i = 1, 2)$, the vectors $r_1 = U_m - U_-$, $r_2 = U_+ - U_m$ form a basis of $\mathbb{R}^2$. Thus, the initial mass can be decomposed into

$$\int_{-\infty}^{\infty}(U(x,0) - \bar{U}^1(x,0))dx = \sum_{i=1}^{2}\alpha_i r_i \tag{1.14}$$

with the uniquely determined constants α_i $(i = 1, 2)$. Now the desired ansatz $\bar{U}^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon}$ is defined as

$$\bar{U}^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon} = \begin{bmatrix} \bar{v}^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon} \\ \bar{u}^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon} \end{bmatrix} = \begin{bmatrix} V_1^\epsilon(x - s_1t + \alpha_1\epsilon) + V_2^\epsilon(x - s_2t + \alpha_2\epsilon) - v_m \\ U_1^\epsilon(x - s_1t + \alpha_1\epsilon) + U_2^\epsilon(x - s_2t + \alpha_2\epsilon) - u_m \end{bmatrix}. \tag{1.15}$$

Then it follows from (1.3), (1.11), (1.12), (1.14) and $\bar{U}^\epsilon(x,t)=\bar{U}^1(\frac{x}{\epsilon},\frac{t}{\epsilon})$ that

$$\begin{aligned}\int_{-\infty}^{\infty}(U(x,0)-\bar{U}^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon}(x,0))dx&=\int_{-\infty}^{\infty}((U(x,0)-\bar{U}^\epsilon(x,0))dx+\int_{-\infty}^{\infty}(\bar{U}^\epsilon(x,0)-\bar{U}^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon}(x,0))dx\\&=\epsilon\sum_{i=1}^{2}\alpha_i r_i-\epsilon\sum_{i=1}^{2}\alpha_i r_i=0.\end{aligned}\tag{1.16}$$

Thus, $\bar{U}^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon}$ is the desired ansatz.

Now, we are in a position to stating our main theorem.

Theorem 1.1. *Let the constant states $U_\pm$ (with $v_\pm>0$) be connected by the composite wave consisting of two shock waves, defined by (1.5) above, and $\delta=|U_+-U_-|$ be sufficiently small. Then the Navier-Stokes equations (1.1) with Riemann initial data (1.3) have a unique, global, piecewise smooth solution $U^\epsilon(x,t)=(v^\epsilon,u^\epsilon)^t(x,t)$ such that:*

(i) *$u^\epsilon(x,t)$ is continuous for $t>0$; u_x^ϵ, v^ϵ, v_x^ϵ and v_t^ϵ are uniformly Hölder continuous in the sets $\{x<0,t\geqslant\tau\}$ and $\{x>0,t\geqslant\tau\}$ for any $\tau>0$; and u_t^ϵ, u_{xx}^ϵ, and v_{xt}^ϵ are Hölder continuous on compact set in $\{(x,t),x\neq0,t>0\}$. Moreover, the jumps in $v^\epsilon(x,t)$ and $u_x^\epsilon(x,t)$ at $x=0$ satisfy*

$$|[v^\epsilon(0,t)]|\leqslant c\exp\{-ct/\epsilon\},\quad |[u_x^\epsilon(0,t)]|\leqslant c\exp\{-ct/\epsilon\},\tag{1.17}$$

where c is a positive constant independent of t and ϵ.

(ii) *The solutions $U^\epsilon=(v^\epsilon,u^\epsilon)^t$ converge uniformly to the composite wave U^0 consisting of two shock waves defined in (1.5) as the viscosity $\epsilon\to0$ on sets of the form $\{(x,t):|x-s_1t|\geqslant h$ and $|x-s_2t|\geqslant h\}$, for any positive number h, i.e.,*

$$\lim_{\epsilon\to0}\sup_{|x-s_it|\geqslant h,i=1,2}|U^\epsilon(x,t)-U^0(x,t)|=0.\tag{1.18}$$

(iii) *For fixed viscosity $\epsilon>0$, the solution $U^\epsilon(x,t)$ approaches the composite wave $\bar{U}^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon}$ consisting of two viscous shock wave defined in (1.15) uniformly as time t goes to infinity, i.e.,*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^1}|U^\epsilon(x,t)-\bar{U}^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon}(x,t)|=0.\tag{1.19}$$

The convergence rate in L^p-distance is given by

$$\sup_{t\geqslant0}\|U^\epsilon(\cdot,t)-U^0(\cdot,t)\|_{L^p}\leqslant C\epsilon^{\frac{1}{p}},\quad \text{for any } 2\leqslant p<\infty,\tag{1.20}$$

where the positive constant C is independent of ϵ and t.

Remark 1.2. It is interesting to make a comparison between Theorem 1.1 and those of Yu [44], where Yu gives a sharp characterization of the zero dissipation limit process with shock and initial layer for the hyperbolic conservation laws with artificial viscosity. The main theorem of [44] is valid on the time interval $\delta^{-2-\alpha_0}\epsilon\leqslant t\leqslant O(1)\delta^3$ (here α_0 is a given positive constant, ϵ and δ denote the viscosity coefficient and the strength of the wave, respectively, see the main theorem on [44, p. 278] for details). The convergence rate in (1.19) and (1.20) is not as good as in [44], but they are valid for all the time $t>0$.

Remark 1.3. Similar method can be applied to study the compressible non-isentropic Navier-Stokes equations. This will be reported in a forthcoming paper [47].

Now, we sketch the main idea of the proof and explain on some of the main difficulties and techniques involved in the process. Roughly speaking, we follow the framework of Hoff and Liu [9] on the case of a single shock wave, and the proof involves the following four steps.

In first step, using the hyperbolic scaling property of (1.1) and (1.3) and the Riemann problems (1.4) and (1.3), we perform the scaling argument to reduce the proof of Theorem 1.1 to the nonlinear stability problem in large time. Therefore, we encounter the problem to prove large time noninear asymptotic stability of a composite wave of two viscous shock waves for (1.1) under the Riemann data (1.3). It is worth mentioning here that for Riemann initial data (1.3), the L^2-norm of the spatial antiderivative of the initial perturbation, i.e., $\int_{-\infty}^{x}(U(y,0)-\bar{U}_{\alpha_1,\alpha_2}(y,0))dy$, is of the order $\delta^{-1/2}$, where $\delta=|U_+-U_-|$. Thus, if we take δ small, the H^2-norm of $\int_{-\infty}^{x}(U(y,0)-\bar{U}_{\alpha_1,\alpha_2}(y,0))dy$ becomes arbitrarily large. Therefore, the classic energy methods in [17,20,26–29,39,40,45], depending essentially on the smallness of the H^2-norm of the spatial antiderivative of the initial perturbation, do not work here. Comparing with the single shock wave case in [9], where the excess mass is zero, the main difficulties here lie in the non-zero excess mass and the interactions of two shock waves which are listed as follows. First, due to non-zero excess mass, we need introduce phase shifts to carry the excessive mass, which makes the problem become much more difficult (see the proofs of (2.12), Lemmas 3.1 and 3.2). We remark here that the shock speeds $s_1<0$ and $s_2>0$ play an essential role in the proof of Lemmas 3.1 and 3.2. Second, due to the interactions of two shock waves, we need to control the error estimates arising from the difference between the two intermediates described in Figures 1 and 2. One of key observations in this paper is that the difference is on the order of δ^3 (see (2.12)). With this key estimates in hand, we can control the error estimates arising from the difference to close our energy estimates. Third, since the composite wave is considered here, we can only get that the difference between $\lambda_2(U_m)$ and $\lambda_2(U_+)$ is on the order of δ, and do not know whether or not it is on the order of δ^3 as in [9]. But, as we know, the fact that the difference between $\lambda_2(U_m)$ and $\lambda_2(U_+)$ is on the order of δ^3 plays an important role in the proof of the main theorem in [9]. We overcome this difficulty by estimating the terms much more carefully and technically (see (2.32), (2.34), Lemmas 2.6 and 3.1). Fourth, since two viscous shock waves and the effects of the initial layers are considered here, wave interactions do occur. By making full use of the underlying wave structure, we can obtain our desired estimates of wave interactions to close our energy estimates (see the proof of Lemma 5.1). Fifth, we need to tackle the difficulty arising from the fact that the composite of two viscous shock waves defined in (1.15) is not the exact solution to the Navier-Stokes equations (1.1) with Riemann initial data (1.3). We overcome this difficulty by combining wave interactions estimates and the technique on energy estimates.

In the second step, we develop sharp approximate solution to Navier-Stokes equations with Riemann data (1.3) in the initial time regime. It is well known that viscous shock waves are leading asymptotic ansatz for shock wave in large time, but do not generate good approximation in short initial time. Therefore, both [9] and us encounter difficulty from initial layer. To overcome this difficulty, we construct approximate solutions through nonlinear Burgers' equation. The key idea is that instead of viscous shock wave, we decompose the Riemann data in phase space and reconnect them through two diffusion waves which are the Navier-Stokes' correspondences through nonlinear Burgers' equation. These give a much better approximation to the Navier-Stokes solution in its leading order and matches well the initial Riemann data. Therefore, detailed local information on the solution is obtained, and the solution is extended to the intermediate time regime of order $O(\delta^{-2-\vartheta})$, where δ denotes the strength of the initial jumps, and ϑ is a small positive constant.

In the third step, we establish the key property of the solution to Navier-Stokes in the critical intermediate time regime. By making full use of the nonlinearity of Burgers' equation and delicate energy methods, we can show the difference between the solution to the Navier-Stokes equations and the approximate solution remains small, at least for times up to intermediate time of order $O(\delta^{-2-\vartheta})$. It is now we are able to deal with the problem caused by the fact that the L^2-norm of the spatial antiderivative of the initial perturbation is as large as $\delta^{-1/2}$. In fact, motivated by [9], one of key observations is that the square of the L^2-norm of $\int_{-\infty}^{x}(U(y,t)-\bar{U}_{\alpha_1,\alpha_2}(y,t))dy$ is of the order $\delta^a(t+1)^b$ (where a and b are nonnegative and $a-2b\geqslant 0$, see (2.63)), which may be arbitrarily large if the strength of the initial jumps δ is sufficiently small and $t=O(\delta^{-2-\vartheta})$. The estimate (2.64) will enable us to obtain that the square of the L^2-norms of higher-order derivatives are of the order $\delta^a(t+1)^b$ (where a and b are nonnegative and $a-2b\geqslant 1$), which may be arbitrarily small if the strength of the initial jumps δ is sufficiently small and $t=O(\delta^{-2-\vartheta})$. This, in turn, will lead to the desired smallness of the L^∞-norm of $\int_{-\infty}^{x}(U(y,t)-\bar{U}_{\alpha_1,\alpha_2}(y,t))dy$ and L^2-norms of higher-order derivatives which is exactly the a priori assumption of Lemma 2.6. The energy estimates thus can be closed. We remark that the smallness assumption on the strength of the initial jumps is essential here.

In the last step, we show that for very large time, the solution to the Navier-Stokes equations coalesces with the composite viscous traveling wave of the Navier-Stokes equations. This argument is proved by means of energy estimates, using time $O(\delta^{-2-\vartheta})$ as initial time. The detailed estimate of solution in intermediate time regime helps to soften the roughness of the initial data due to dissipation of Navier-Stokes. The resumed smallness in certain order norms gives the possibility to prove this stability result using energy method. Comparing with [17], the main novelty in this step of this paper is to overcome the difficulties arising from non-smooth initial perturbations and the careful energy estimate on the boundary integral terms. These can be easily seen from the new and very different energy estimates in, for instance, the proofs of (5.2) and (5.3), and the estimates on the boundary integral terms arising from the non-smooth initial perturbations (see (5.20)). For other related works, we refer the readers to [32–36, 38].

The rest of this paper is organized as follows. In the next section, we construct a certain approximate solution and collect together those properties needed for energy estimates in Sections 4–5. In Section 3, we collect some fundamental facts concerning the viscous shock waves. The wave interactions are also estimated in this section. In Section 4, we obtain the intermediate-time estimate for $U - \bar{U}_{\alpha_1,\alpha_2}$. In Section 5, we make careful energy estimates to complete the proof of our main results, Theorem 1.1.

Throughout this paper, we use the following notations:

$$\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}, \quad \|\cdot\Vert\!\!\!\!\!\;+ = \|\cdot\|_{L^2(\mathbb{R}^-)} + \|\cdot\|_{L^2(\mathbb{R}^+)}, \quad \fint dy = \int_{-\infty}^{0} dy + \int_{0}^{+\infty} dy.$$

2 Some properties for the approximate solution

In this section, we adjust the technique developed by Hoff and Liu [9] to construct the approximate solution based on the self-similar solutions of the Burgers equation and collect together some estimates needed in Sections 3–5. We shall take $\epsilon = 1$ throughout this section, so that (1.1) takes the form

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \left(\dfrac{u_x}{v}\right)_x. \end{cases} \tag{2.1}$$

Rewrite (2.1) as

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}\left(B(U)\frac{\partial U}{\partial x}\right), \tag{2.2}$$

where

$$U = \begin{pmatrix} v \\ u \end{pmatrix}, \quad A(U) = \begin{pmatrix} 0, & -1 \\ p'(v), & 0 \end{pmatrix}, \quad B(U) = \begin{pmatrix} 0, & 0 \\ 0, & \frac{1}{v} \end{pmatrix}.$$

The characteristic speeds λ_i and right eigenvectors r_i, $i = 1, 2$, for A are

$$\lambda_i = \mp\sqrt{\sigma}, \quad r_i = \begin{bmatrix} \mp 1/\sigma_v \\ -\sigma/\sigma_v \end{bmatrix}, \quad \nabla\lambda_i \cdot r_i \equiv 1, \quad i = 1, 2, \tag{2.3}$$

where $\sigma = \sqrt{-p'(v)}$.

Let R_1 and R_2 be integral curves of r_1 and r_2, passing through U_- and U_+, respectively, and intersecting at $\tilde{U}_m$ (see Figure 2).

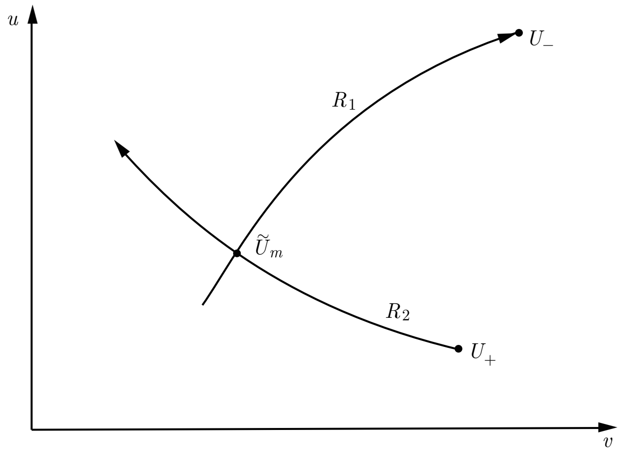

Figure 2 The approximate diffusion waves

It is noted that the Rankine-Hugoniot condition (1.4) implies

$$u_- = u_+ + \sqrt{(\tilde{v}_m - v_-)(p(v_-) - p(\tilde{v}_m))} + \sqrt{(\tilde{v}_m - v_+)(p(v_+) - p(\tilde{v}_m))} \tag{2.4}$$

and the integral curves R_i, $i = 1, 2$, are given by

$$R_1 : u = u_- - \int_v^{v_-} \sqrt{-p'(\tau)}d\tau, \quad R_2 : u = u_+ - \int_{v_+}^{v} \sqrt{-p'(\tau)}d\tau. \tag{2.5}$$

By virtue of (2.4)–(2.5) and Hölder inequality, we have

$$\begin{aligned} 2(\tilde{u}_m - u_+) &= \sqrt{(\tilde{v}_m - v_-)(p(v_-) - p(\tilde{v}_m))} \\ &\quad + \sqrt{(\tilde{v}_m - v_+)(p(v_+) - p(\tilde{v}_m))} - \int_{v_+}^{v_-} \sqrt{-p'(\tau)}d\tau \\ &= \sqrt{(\tilde{v}_m - v_-)(p(v_-) - p(\tilde{v}_m))} - \int_{\tilde{v}_m}^{v_-} \sqrt{-p'(\tau)}d\tau \\ &\quad + \sqrt{(\tilde{v}_m - v_+)(p(v_+) - p(\tilde{v}_m))} - \int_{v_+}^{\tilde{v}_m} \sqrt{-p'(\tau)}d\tau \\ &\geqslant 0. \end{aligned} \tag{2.6}$$

Therefore, we see that $\tilde{v}_m < v_+$ and $\lambda_2(\tilde{U}_m) > \lambda_2(U_+)$, since $\sigma_v < 0$. So, we define

$$\tilde{\delta}_1 = \frac{1}{2}[\lambda_1(U_-) - \lambda_1(\tilde{U}_m)] > 0 \tag{2.7}$$

and

$$\tilde{\delta}_2 = \frac{1}{2}[\lambda_2(\tilde{U}_m) - \lambda_2(U_+)] > 0. \tag{2.8}$$

It follows from (1.8) , triangle inequality and $|\tilde{U}_m - U_-| + |U_+ - \tilde{U}_m| \leqslant C|U_+ - U_-|$ that

$$\delta \leqslant \tilde{\delta}_1 + \tilde{\delta}_2 \leqslant C\delta. \tag{2.9}$$

By virtue of [37, Theorem 17.16], we have

$$\begin{aligned} &\left| u_m - \int^{v_m} \sqrt{-p'(\tau)}d\tau - u_- + \int^{v_-} \sqrt{-p'(\tau)}d\tau \right| \\ &\quad = \left| u_m - \int^{v_m} \sqrt{-p'(\tau)}d\tau - \tilde{u}_m + \int^{\tilde{v}_m} \sqrt{-p'(\tau)}d\tau \right| \leqslant C|U_m - U_-|^3 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned}&\left|u_m+\int^{v_m}\sqrt{-p'(\tau)}d\tau-u_+-\int^{v_+}\sqrt{-p'(\tau)}d\tau\right|\\&\quad=\left|u_m+\int^{v_m}\sqrt{-p'(\tau)}d\tau-\tilde{u}_m-\int^{\tilde{v}_m}\sqrt{-p'(\tau)}d\tau\right|\leqslant C|U_+-U_m|^3.\end{aligned}\tag{2.11}$$

From (1.9), (2.10) and (2.11), we can get the desired estimate concerning the difference $|U_m-\tilde{U}_m|$, i.e.,

$$|U_m-\tilde{U}_m|\leqslant C\delta^3.\tag{2.12}$$

Using (1.9), (1.10), (2.9), (2.12) and triangle inequality, we have

$$\delta/C\leqslant\tilde{\delta}_1,\quad \tilde{\delta}_2\leqslant C\delta.\tag{2.13}$$

Now, we are ready to describe Riemann data solutions and traveling wave solutions to the Burgers equation, and use this information to construct the approximations $\bar{U}_R$ and $\bar{U}_{\rm TW}$ required later on. To begin with, we let λ_R be the solution to the Burgers equation

$$\lambda_t+\lambda\lambda_x=\beta\lambda_{xx},\quad t>0,\quad x\in\mathbb{R},\tag{2.14}$$

with Riemann initial data

$$\lambda_R(x,0)=\begin{cases}0, & x<0,\\ -2\hat{\delta}, & x>0,\end{cases}\tag{2.15}$$

where β and $\hat{\delta}$ are positive constants.

Using the well-known Hopf-Cole transform, we can solve initial value problems (2.14) and (2.15) directly. As in [9], we have the following lemma.

Lemma 2.1. *The solution to initial value problems* (2.14) *and* (2.15) *is given by*

$$\lambda_R=-2\hat{\delta}\frac{\mathrm{e}^{\hat{\delta}(x+\hat{\delta}t)/\beta}f(-\frac{x+2\hat{\delta}t}{\sqrt{4\beta t}})}{\mathrm{e}^{\hat{\delta}(x+\hat{\delta}t)/\beta}f(-\frac{x+2\hat{\delta}t}{\sqrt{4\beta t}})+f(\frac{x}{\sqrt{4\beta t}})},\tag{2.16}$$

where

$$f(a)=\pi^{-\frac{1}{2}}\int_a^{+\infty}\mathrm{e}^{-\tau^2}d\tau.\tag{2.17}$$

Moreover, λ_R *satisfies*

$$-2\hat{\delta}\leqslant\lambda_R\leqslant 0,\tag{2.18}$$

$$\begin{cases}\left|\dfrac{\partial\lambda_R}{\partial x}(x,t)\right|\leqslant C[\hat{\delta}t^{-\frac{1}{2}}\mathrm{e}^{-\frac{x^2}{4\beta t}}+\hat{\delta}^2\mathrm{e}^{-\hat{\delta}|x+\hat{\delta}t|/\beta}],\\ \left|\dfrac{\partial^2\lambda_R}{\partial x^2}(x,t)\right|\leqslant C[(\hat{\delta}|x|t^{-\frac{3}{2}}+\hat{\delta}^2t^{-\frac{1}{2}})\mathrm{e}^{-\frac{x^2}{4\beta t}}+\hat{\delta}^3\mathrm{e}^{-\hat{\delta}|x+\hat{\delta}t|/\beta}],\\ \left|\dfrac{\partial^3\lambda_R}{\partial x^3}(x,t)\right|\leqslant C\{[\hat{\delta}(x^2t^{-3}+|x|t^{-\frac{5}{2}})+\hat{\delta}^2(|x|t^{-\frac{3}{2}}+t^{-1})+\hat{\delta}^3t^{-\frac{1}{2}}]\mathrm{e}^{-\frac{x^2}{4\beta t}}+\hat{\delta}^4\mathrm{e}^{-\hat{\delta}|x+\hat{\delta}t|/\beta}]\}.\end{cases}\tag{2.19}$$

Proof. See [9, Theorem 2.1]. □

So, we let λ_R^i, $i=1,2$, be the solution to the Burgers equation (2.14) with initial data

$$\lambda_R^i(x,0)=\begin{cases}0, & x<0,\\ -2\tilde{\delta}_i, & x>0,\end{cases}\tag{2.20}$$

and $\beta = 1/(2\tilde{v}_m)$, and set

$$\begin{cases} \mu_R^1(x,t) = \lambda_R^1(x - \lambda_1(U_-)(t+1), t+1) + \lambda_1(U_-), \\ \mu_R^2(x,t) = \lambda_R^2(x - \lambda_2(\tilde{U}_m)(t+1), t+1) + \lambda_2(\tilde{U}_m). \end{cases} \tag{2.21}$$

It is easy to see that μ_i are also solutions to (2.14) with "initial data"

$$\mu_R^1(x,-1) = \begin{cases} \lambda_1(U_-), & x < 0, \\ \lambda_1(\tilde{U}_m), & x > 0, \end{cases}$$

and

$$\mu_R^2(x,-1) = \begin{cases} \lambda_2(\tilde{U}_m), & x < 0, \\ \lambda_2(U_+), & x > 0. \end{cases}$$

Next, we generate approximate solutions $\bar{U}_R^i$ by taking $\bar{U}_R^i \in R_i$ and $\lambda_i(\bar{U}_R^i(x,t)) = \mu_R^i(x,t)$, and we finally set

$$\bar{U}_R = \bar{U}_R^1 + \bar{U}_R^2 - \tilde{U}_m. \tag{2.22}$$

Then, $\bar{U}_R$ satisfies the pde

$$(\bar{U}_R)_t + F(\bar{U}_R)_x = B(\tilde{U}_m)(\bar{U}_R)_{xx} - A_1 - A_2 + D_x, \tag{2.23}$$

where

$$A_i = \sum_{j \neq i} \frac{1}{2\tilde{v}_m} (\mu_R^i)_{xx} r_j(\bar{U}_R^i) + (\mu_R^i)_x B(\tilde{U}_m)(r_i(\bar{U}_R^i))_x, \tag{2.24}$$

and

$$F(U) = \begin{bmatrix} -u \\ p(v) \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ p(\bar{v}_R) - p(\bar{v}_R^1) - p(\bar{v}_R^2) + p(\tilde{v}_m) \end{bmatrix}. \tag{2.25}$$

At $t = -1$, $\bar{U}_R$ agrees with $U(\cdot, 0)$,

$$\bar{U}_R(x,-1) = \begin{cases} U_-, & x < 0, \\ U_+, & x > 0. \end{cases} \tag{2.26}$$

An approximate solution $\bar{U}_{\mathrm{TW}}$ is constructed in exactly the same way, except that, in place of the solutions λ_R^i to the Burgers equation with Riemann data, we substitute the corresponding traveling wave with the traveling wave data $\lambda_{\mathrm{TW}}^i(x,0) = (-2\tilde{\delta}_i)/(1 + \mathrm{e}^{-\tilde{\delta}_i x/\beta})$ with $\beta = 1/(2\tilde{v}_m)$. Thus,

$$\lambda_{\mathrm{TW}}^i(x,t) = \frac{-2\tilde{\delta}_i}{1 + \mathrm{e}^{-\tilde{\delta}_i (x - \tilde{s}_i t)/\beta}}, \tag{2.27}$$

where

$$\tilde{s}_1 = \frac{\lambda_1(U_-) + \lambda_1(\tilde{U}_m)}{2}, \quad \tilde{s}_2 = \frac{\lambda_2(\tilde{U}_m) + \lambda_2(\tilde{U}_+)}{2}, \tag{2.28}$$

and

$$\begin{cases} \mu_{\mathrm{TW}}^1(x,t) = \lambda_{\mathrm{TW}}^1(x,t+1) + \lambda_1(U_-), \\ \mu_{\mathrm{TW}}^2(x,t) = \lambda_{\mathrm{TW}}^2(x,t+1) + \lambda_2(\tilde{U}_m), \\ \bar{U}_{\mathrm{TW}} = \bar{U}_{\mathrm{TW}}^1 + \bar{U}_{\mathrm{TW}}^2 - \tilde{U}_m, \\ \lambda_i(\bar{U}_{\mathrm{TW}}^i(x,t)) = \mu_{\mathrm{TW}}^i(x,t), \quad \bar{U}_{\mathrm{TW}}^i \in R_i. \end{cases} \tag{2.29}$$

In the following lemma, we collect together those properties for $\bar{U}_R$ needed in Sections 3–4.

Lemma 2.2. *Let $\bar{U}_R$ be as constructed above. Then*

$$\left\|\frac{\partial \bar{U}_R}{\partial x}(\cdot,t)\right\| \leqslant C[\delta(t+1)^{-1/4}+\delta^{3/2}], \tag{2.30}$$

$$\int_0^t\int_R\left(\frac{\partial \bar{U}_R}{\partial x}\right)^2 dxdt \leqslant C[\delta^2(t+1)^{1/2}+\delta^3 t], \tag{2.31}$$

$$\left\|\frac{\partial \bar{U}_R}{\partial t}(\cdot,t)\right\| \leqslant C[\delta(t+1)^{-3/4}+\delta^{2}], \tag{2.32}$$

$$\left\|\frac{\partial \bar{U}_R}{\partial x}(\cdot,t)\right\|_{L^\infty},\quad \left\|\frac{\partial \bar{U}_R}{\partial t}(\cdot,t)\right\|_{L^\infty} \leqslant C[\delta(t+1)^{-1/2}+\delta^{2}], \tag{2.33}$$

$$\left\|\frac{\partial \bar{U}_R}{\partial x}(\cdot,t)\right\|_{L^1} \leqslant C\delta, \tag{2.34}$$

$$\frac{\partial \bar{v}_R}{\partial t} = f_1+f_2, \tag{2.35}$$

where $-C\delta\leqslant f_1<0$, *and* $\|f_2(\cdot,t)\|_{L^\infty}\leqslant C[\delta(t+1)^{-1}+\delta^2(t+1)^{-1/2}+\delta^3]$,

$$\int_0^{+\infty}\left|\int_x^{\infty}(\bar{U}_R^1-\tilde{U}_m)(y,t)dy\right|^2 dx \leqslant C\delta^2(t+1)^{3/2}\mathrm{e}^{-(t+1)/C}, \tag{2.36}$$

$$\int_{-\infty}^{0}\left|\int_{-\infty}^{x}(\bar{U}_R^2-\tilde{U}_m)(y,t)dy\right|^2 dx \leqslant C\delta^2(t+1)^{3/2}\mathrm{e}^{-(t+1)/C}, \tag{2.37}$$

$$\|\bar{U}_R(\cdot,0)-\bar{U}_R(\cdot,-1)\| \leqslant C\delta, \tag{2.38}$$

$$\int_0^{+\infty}\int_R|D|dxdt \leqslant C\delta^2,\quad \int_0^{+\infty}\int_R|D|^2dxdt \leqslant C\delta^4. \tag{2.39}$$

Proof. Since the proofs of (2.30)–(2.32), (2.34) and (2.36)–(2.39) can be found in [9], we focus our attention on the proofs of (2.33) and (2.35). To begin with, we have from construction that

$$\begin{aligned}\left|\frac{\partial \bar{U}_R}{\partial x}(x,t)\right| &\leqslant \left|\frac{\partial \mu_R^1}{\partial x}(x,t)r_1(\bar{U}_R^1)\right| + \left|\frac{\partial \mu_R^2}{\partial x}(x,t)r_2(\bar{U}_R^2)\right|\\ &\leqslant C\left|\frac{\partial \lambda_R^1}{\partial x}(x-\lambda_1(U_-)(t+1),t+1)\right| + \left|\frac{\partial \lambda_R^2}{\partial x}(x-\lambda_2(\tilde{U}_m)(t+1),t+1)\right|\\ &\leqslant C[\delta(t+1)^{-1/2}+\delta^2],\end{aligned} \tag{2.40}$$

where we have used (2.13) and (2.19).

Similarly, we can obtain

$$\left|\frac{\partial \bar{U}_R}{\partial t}(x,t)\right| \leqslant C[\delta(t+1)^{-1/2}+\delta^2]. \tag{2.41}$$

Then, (2.33) follows from (2.40) and (2.41) immediately.

Next, we prove (2.35). By construction, $\frac{\partial \bar{U}_R^i}{\partial t}=(\mu_R^i)_t r_i(\bar{U}_R^i)$, $i=1,2$, so that by (2.3)

$$\frac{\partial \bar{v}_R}{\partial t} = a_1\frac{\partial \mu_R^1}{\partial t} - a_2\frac{\partial \mu_R^2}{\partial t}, \tag{2.42}$$

where $a_i=O(1)$, $i=1,2$, are positive.

By virtue of (2.22), we have

$$\frac{\partial \bar{v}_R^1}{\partial t} = \left[\frac{\partial \lambda_R^1}{\partial t}(x-\lambda_1(U_-)(t+1),t+1) - \lambda_1(U_-)\frac{\partial \lambda_R^1}{\partial x}(x-\lambda_1(U_-)(t+1),t+1)\right], \tag{2.43}$$

and

$$\frac{\partial \bar{v}_R^2}{\partial t} = \left[\frac{\partial \lambda_R^2}{\partial t}(x-\lambda_1(\tilde{U}_m)(t+1),t+1) - \lambda_2(\tilde{U}_m)\frac{\partial \lambda_R^2}{\partial x}(x-\lambda_2(\tilde{U}_m)(t+1),t+1)\right]. \tag{2.44}$$

Combining the above relations (2.42)–(2.44), we get

$$\frac{\partial \bar{v}_R}{\partial t} = \left[-a_1\lambda_1(U_-)\frac{\partial \lambda_R^1}{\partial x}(x-\lambda_1(U_-)(t+1),t+1) + a_2\lambda_2(\tilde{U}_m)\frac{\partial \lambda_R^2}{\partial x}(x-\lambda_2(\tilde{U}_m)(t+1),t+1)\right] + \left[a_1\frac{\partial \lambda_R^1}{\partial t}(x-\lambda_1(U_-)(t+1),t+1) - a_2\frac{\partial \lambda_R^2}{\partial t}(x-\lambda_2(\tilde{U}_m)(t+1),t+1)\right]. \tag{2.45}$$

The first term here, which we take to be f_1 is negative, since the solution operator to (2.14) preserves monotonicity. It is clear that $-C\delta \leqslant f_1 \leqslant 0$. The second term above, which we define to be f_2, then satisfies

$$\|f_2(\cdot,t)\|_{L^\infty} \leqslant C[\delta(t+1)^{-1} + \delta^2(t+1)^{-1/2} + \delta^3], \tag{2.46}$$

where we have used (2.13), (2.14) and (2.18).

Therefore, the proof of (2.35) is completed. □

The following facts concerning $\bar{U}_{\mathrm{TW}}$ will be needed in Sections 3–4.

Lemma 2.3. *Let $\bar{U}_{\mathrm{TW}}$ be as constructed above. Then*

$$\left\|\frac{\partial}{\partial x}\bar{U}^i_{\mathrm{TW}}(\cdot,t)\right\| \leqslant C\delta^{3/2}, \tag{2.47}$$

and

$$\|\bar{U}_{\mathrm{TW}}(\cdot,0) - \bar{U}_{\mathrm{TW}}(\cdot,-1)\| \leqslant C\delta^{3/2}. \tag{2.48}$$

Proof. See [9, Theorem 2.5]. □

The following lemma is concerned with the estimates of the difference between $\bar{U}_R$ and $\bar{U}_{\mathrm{TW}}$.

Lemma 2.4. *Let $\bar{U}_R$ and $\bar{U}_{\mathrm{TW}}$ be as constructed above. Then*

$$\|\bar{U}_R(\cdot,t) - \bar{U}_{\mathrm{TW}}(\cdot,t)\| \leqslant C[\delta^{1/2} + \delta(t+1)^{1/4}]\mathrm{e}^{-\delta^2(t+1)/C}, \tag{2.49}$$

$$\int_{-\infty}^0 \left|\int_{-\infty}^x (\bar{U}_R^1 - \bar{U}_{\mathrm{TW}}^1)(y,t)dy\right|^2 dx \leqslant C[\delta^{-1/2} + \delta(t+1)^{1/4}]\mathrm{e}^{-\delta^2(t+1)/C}, \tag{2.50}$$

$$\int_0^{\infty} \left|\int_x^{\infty} (\bar{U}_R^2 - \bar{U}_{\mathrm{TW}}^2)(y,t)dy\right|^2 dx \leqslant C[\delta^{-1/2} + \delta(t+1)^{1/4}]\mathrm{e}^{-\delta^2(t+1)/C}. \tag{2.51}$$

Proof. Noting that (2.13), we can prove Lemma 2.3 by applying the similar arguments as in [9, Theorem 2.6]. □

The approximate solution $\bar{U}_R$ constructed above fails to satisfy a conservation equation (see (2.23)), so that the variable

$$\int_{-\infty}^x [U(y,t) - \bar{U}_R(y,t)]dy$$

is not necessarily in $L^2(\mathbb{R})$ for $t>0$. We therefore introduce the function $\bar{\bar{U}}$ defined by

$$\bar{\bar{U}}_t + F'(\tilde{U}_m)\bar{\bar{U}}_x = \frac{1}{2\tilde{v}_m}\bar{\bar{U}}_{xx} + A_1 + A_2, \quad x\in\mathbb{R}, \quad t>0, \tag{2.52}$$

with the initial data

$$\bar{\bar{U}}(x,0) = \bar{U}_R(x,-1) - \bar{U}_R(x,0). \tag{2.53}$$

Thus, $\bar{U}_R + \bar{\bar{U}}$ agrees with U at $t=0$ (see (2.26)), and $U - \bar{U}_R - \bar{\bar{U}}$, satisfying a conservation equation, will have an L^2 x-antiderivative for all time.

The following lemma gives some bounds for $\bar{\bar{U}}$ required for the analysis in Sections 3–4.

Lemma 2.5. *Let $\bar{\bar{U}}$ be constructed above , and assume that $\delta = |U_+ - U_-|$ is sufficiently small. Then*

$$\|\bar{\bar{U}}(\cdot,t)\| \leqslant C \sum_{a-2b\geqslant 1} \delta^a (t+1)^b, \tag{2.54}$$

$$\int_0^t \int_R |\bar{\bar{U}}|^2 dx dt \leqslant C \sum_{a-2b\geqslant 0} \delta^a (t+1)^b, \tag{2.55}$$

$$\int_0^t \int_R |\bar{\bar{U}}_x|^2 dx dt \leqslant C \sum_{a-2b\geqslant 2} \delta^a (t+1)^b, \tag{2.56}$$

$$\|\bar{\bar{U}}_x(\cdot,t)\| \leqslant C \sum_{a-2b\geqslant 1} \delta^a (t+1)^b, \quad t \geqslant 1, \tag{2.57}$$

$$\int_1^t \int_R |\bar{\bar{U}}_{xx}|^2 dx dt \leqslant C \sum_{a-2b\geqslant 2} \delta^a (t+1)^b, \quad t \geqslant 1, \tag{2.58}$$

$$\int_1^t \int_R |\bar{\bar{U}}|^4 dx dt \leqslant C \sum_{a-2b\geqslant 2} \delta^a (t+1)^b, \quad t \geqslant 1, \tag{2.59}$$

$$\|\bar{\bar{U}}_t(\cdot,t)\| \leqslant C\delta, \quad t \geqslant 1, \tag{2.60}$$

$$\int_{-\infty}^0 \left| \int_{-\infty}^x \bar{\bar{U}}(y,t) dy \right|^2 dx \leqslant C \sum_{a-2b\geqslant 0} \delta^a (t+1)^b, \tag{2.61}$$

$$\int_0^{\infty} \left| \int_x^{\infty} \bar{\bar{U}}(y,t) dy \right|^2 dx \leqslant C \sum_{a-2b\geqslant 0} \delta^a (t+1)^b. \tag{2.62}$$

Here, $\sum_{a-2b\geqslant c} \delta^a (t+1)^b$ denotes a finite sum of the terms of the form $\delta^a (t+1)^b$ where a and b are nonnegative and $a - 2b \geqslant c$.

Proof. See [9, Theorem 3.1]. □

We end this section with the following lemma concerning the existence of the solution U up to the intermediate time $T = O(\delta^{-2-\vartheta})$:

Lemma 2.6. *Under the assumptions of Theorem 1.1, the solution to (1.3)–(2.1) exists up to the intermediate time $T = O(\delta^{-2-\vartheta})$, where $\vartheta > 0$ is a global constant, and satisfies*

$$\sup_{0\leqslant\tau\leqslant t} \|W(\tau)\|^2 + \int_0^t \|W_x(\tau)\|^2 d\tau + \int_0^t \int_R |f_1| w^2 dx d\tau \leqslant C \sum_{a-2b\geqslant 0} \delta^a (t+1)^b, \tag{2.63}$$

$$\sup_{0\leqslant\tau\leqslant t} \{ \|\Delta U(\tau)\|\!\!\!\!\,|^2 + \|v_x(\tau)\|\!\!\!\!\,|^2 \} + \int_0^t \|\Delta U_x(\tau)\|\!\!\!\!\,|^2 d\tau \leqslant C \sum_{a-2b\geqslant 1} \delta^a (t+1)^b, \tag{2.64}$$

for $t \leqslant T$. Here $\Delta U = U - \bar{U}_R - \bar{\bar{U}}$ and $W = \int_{-\infty}^x \Delta U(y,t) dy = [\,{}^z_w\,]$.

Proof. Setting $\Delta U = U - \bar{U}_R - \bar{\bar{U}}$ and $W = \int_{-\infty}^x \Delta U(y,t) dy = [\,{}^z_w\,]$, we have

$$\begin{aligned} &\Delta U_t + [F(U) - F(\bar{U}_R) - F'(\tilde{U}_m)\bar{\bar{U}}]_x \\ &\quad = B(\tilde{U}_m)\Delta U_{xx} + [(B(U) - B(\tilde{U}_m))U_x]_x + (B(\tilde{U}_m) - \beta)\bar{\bar{U}}_{xx} - D_x, \end{aligned} \tag{2.65}$$

and

$$W_t + [F(U) - F(\bar{U}_R) - F'(\tilde{U}_m)\bar{\bar{U}}] = B(\tilde{U}_m) W_{xx} + (B(U) - B(\tilde{U}_m))U_x + (B(\tilde{U}_m) - \beta)\bar{\bar{U}}_x - D. \tag{2.66}$$

Of course, the initial data is identically zero for both ΔU and W by (2.26) and (2.53). Linearizing the second equation in (2.66), we can rewrite (2.66) in the form

$$\begin{cases} z_t - w_x = -\beta \bar{\bar{v}}_x, \\ w_t + p'(\bar{v}_R) z_x = \dfrac{w_{xx}}{\tilde{v}_m} + \left(\dfrac{1}{v} - \dfrac{1}{\tilde{v}_m}\right) u_x + \left(\dfrac{1}{\tilde{v}_m} - \beta\right) \bar{\bar{u}}_x - d + O[(v - \bar{v}_R)^2 + |\bar{\bar{v}}(\bar{v}_R - \tilde{v}_m)|]. \end{cases} \tag{2.67}$$

Following the arguments in [9], in order to complete the proof of Lemma 2.6, we only need to prove (2.63) and (2.64) hold up to T as a priori bounds, that is, provided that $\|w\|_{L^\infty}$, $\|v-v_-\|_{L^\infty}$, $\delta^3 t$ and $\delta\log(t+1)$ are sufficiently small. First, we derive energy estimates for the solution W to (2.67). By multiplying the two equations (2.67) by z and $-w/p'(\bar v_R)$, respectively and adding and integrating, we obtain

$$\begin{aligned}
&\sup_{0\leqslant\tau\leqslant t}\int_R |W(x,\tau)|^2\,dx+\int_0^t\int_R(|f_1|w^2+w_x^2)dxd\tau\\
&\quad\leqslant C\int_0^t\int_R[|f_2|w^2+|z_x\bar{\bar v}|+|ww_x(\bar v_R)_x|+|wu_x(v-\tilde v_m)|\\
&\qquad+(|w_x|+|w(\bar v_R)_x|)|\bar{\bar u}|+|w||d|+|w||v-\bar v_R|^2+|w\bar{\bar v}(\bar v_R-\tilde v_m)|]dxd\tau\\
&\quad=\sum_{j=1}^8 I_j,
\end{aligned}\tag{2.68}$$

where we have used the fact that $(\frac{1}{p'(\bar v_R)})_t=|\frac{p''}{(p')^2}f_1|+O(f_2)$.

Now, we apply the estimates (2.30)–(2.39) for $\bar U_R$ and (2.54)–(2.62) for $\bar{\bar U}$ to bound the terms I_j one by one. First, we have from (2.35) that

$$I_1\leqslant C[\delta\log(t+1)+\delta^2(t+1)^{\frac12}+\delta^3 t]\sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)dx.\tag{2.69}$$

Using the Cauchy-Schwarz inequality and (2.54), we get

$$I_2\leqslant C\bigg[\alpha\int_0^t\int_R z_x^2dx\tau+C_\alpha\sum_{a-2b\geqslant0}\delta^a(t+1)^b)\bigg],\tag{2.70}$$

where α is to be chosen later.

Applying the Cauchy-Schwarz inequality and (2.33), we obtain

$$\begin{aligned}
I_3&\leqslant C\alpha\int_0^t\int_R w_x^2dxd\tau+C_\alpha\int_0^t\int_R(\bar v_R)_x^2w^2dxd\tau\\
&\leqslant C\bigg[\alpha\int_0^t\int_R w_x^2dxd\tau+C_\alpha(\delta^2\log(t+1)+\delta^4 t)\sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)d\tau\bigg].
\end{aligned}\tag{2.71}$$

Due to the triangle inequality, we have

$$I_4\leqslant C\int_0^t\int_R|w(v-\tilde v_m)|\bigg(|\Delta u_x|+\bigg|\frac{\partial\bar u_R}{\partial x}\bigg|+|\bar{\bar u}_x|\bigg)dxd\tau=I_4^1+I_4^2+I_4^3.\tag{2.72}$$

Using the triangle inequality, Cauchy-Schwarz inequality, and (2.55), we get

$$\begin{aligned}
I_4^1&\leqslant C\int_0^t\int_R|w||\Delta u_x|(|z_x|+|(\bar v_R-\tilde v_m)|+|\bar{\bar v}|)dxd\tau\\
&\leqslant C\|w\|_{L^\infty}\int_0^t\int_R(z_x^2+\Delta u_x^2)dxd\tau+C\alpha\sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)dx\\
&\quad+C_\alpha\delta^2 t\int_0^t\int_R\Delta u_x^2dxd\tau+C\int_0^t\int_R\Delta u_x^2dxd\tau+C\int_0^t\int_R\bar{\bar v}^2dxd\tau\\
&\leqslant C\bigg[\alpha\sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)dx+\|w\|_{L^\infty}\int_0^t\int_R z_x^2dxd\tau\\
&\quad+(1+\delta^2 t)\int_0^t\int_R\Delta u_x^2dxd\tau+\sum_{a-2b\geqslant0}\delta^a(t+1)^b\bigg].
\end{aligned}\tag{2.73}$$

Similarly, we have

$$I_4^2 \leqslant C\left[\delta^3 t \sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)dx + \alpha\int_0^t\int_R z_x^2 dxd\tau + \sum_{a-2b\geqslant 0}\delta^a(t+1)^b\right], \tag{2.74}$$

and

$$I_4^3 \leqslant C\left[\delta^3 t \sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)dx + \|w\|_{L^\infty}\int_0^t\int_R z_x^2 dxd\tau + \sum_{a-2b\geqslant 0}\delta^a(t+1)^b\right]. \tag{2.75}$$

Combining the relations (2.72)–(2.75), we obtain

$$\begin{aligned} I_4 \leqslant C\bigg[&(\alpha+\delta^3 t)\sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)dx + (\alpha+\|w\|_{L^\infty})\int_0^t\int_R z_x^2 dxd\tau \\ &+ (1+\delta^2 t)\int_0^t\int_R \Delta u_x^2 dxd\tau + \sum_{a-2b\geqslant 0}\delta^a(t+1)^b\bigg]. \end{aligned} \tag{2.76}$$

By virtue of (2.33), (2.55) and Cauchy-Schwarz inequality, we get

$$\begin{aligned} I_5 &\leqslant C\left[\alpha\int_0^t\int_R w_x^2 dxd\tau + \int_0^t\int_R \bar{\bar{u}}^2 dxd\tau + \int_0^t\int_R w^2(\bar{v}_R)_x^2 dxd\tau\right] \\ &\leqslant C\left[\alpha\int_0^t\int_R w_x^2 dxd\tau + (\delta^2\log(t+1)+\delta^4 t)\sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)dx + \sum_{a-2b\geqslant 0}\delta^a(t+1)^b\right]. \end{aligned} \tag{2.77}$$

Applying Cauchy-Schwarz inequality and $(2.39)_2$, we have

$$I_6 \leqslant C\left[\delta^3\int_0^t\int_R w^2 dxd\tau + \delta^{-3}\int_0^t\int_R d^2 dxd\tau\right] \leqslant C\left[\delta^3 t\sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)dx + \delta\right]. \tag{2.78}$$

By the triangle inequality, Cauchy-Schwarz inequality, and (2.55), we obtain

$$I_7 \leqslant C\int_0^t\int_R |w|(|z_x|+|\bar{\bar{v}}|)^2 dxd\tau \leqslant C\|w\|_{L^\infty}\left[\int_0^t\int_R z_x^2 dxd\tau + \sum_{a-2b\geqslant 0}\delta^a(t+1)^b\right]. \tag{2.79}$$

Similarly, we have

$$I_8 \leqslant C\left[\alpha\sup_{0\leqslant\tau\leqslant t}\int_R w^2(x,\tau)dx + \sum_{a-2b\geqslant 0}\delta^a(t+1)^b\right]. \tag{2.80}$$

Inserting the above estimates into (2.68), we get

$$\begin{aligned} &\sup_{0\leqslant\tau\leqslant t}\int_R |W(x,t)|^2 dx + \int_0^t\int_R (|f_1|w^2 + w_x^2)dxd\tau \\ &\quad\leqslant C(\alpha+\|w\|_{L^\infty})\int_0^t\int_R z_x^2 dxd\tau + C(1+\delta^2 t)\int_0^t\int_R \Delta u_x^2 dxd\tau + C\sum_{a-2b\geqslant 0}\delta^a(t+1)^b. \end{aligned} \tag{2.81}$$

Next, we estimate the term $\int_0^t\int_R z_x^2 dxd\tau$. To do this, we differentiate the first equation in (2.67) with respect to x, multiply by $-w$, and add to z_x times the second equation in (2.67). Integrating the resultant equation over $R\times(0,t)$ and applying the similar arguments just as before, we have

$$\int_0^t\int_R z_x^2 dxd\tau \leqslant C\left[\int_R (z_x^2(x,t)+w^2(x,t))dx + \int_0^t\int_R (w_x^2+\Delta u_x^2)dxd\tau + \sum_{a-2b\geqslant 0}\delta^a(t+1)^b\right]. \tag{2.82}$$

Adding a small multiple of (2.82) to (2.81), taking α small in (2.81), and using the priori assumption $\|w\|_{L^\infty}$ is small, we conclude that

$$\sup_{0\leqslant\tau\leqslant t}\int_R |W(x,t)|^2 dx + \int_0^t\int_R (|f_1|w^2 + w_x^2)dxd\tau$$

$$\leqslant C\left[\sup_{0\leqslant\tau\leqslant t}\int_R z_x^2(x,\tau)dx+\sum_{a-2b\geqslant 0}\delta^a(t+1)^b+(1+\delta^2 t)\int_0^t\int_R \Delta u_x^2 dxd\tau\right]. \tag{2.83}$$

With the essential estimate (2.83) in hand, we can use the same arguments as in [9] to conclude that (2.63) and (2.64) hold up to T as a priori bounds, i.e., provided that $\|w\|_{L^\infty}$, $\|v-v_-\|_{L^\infty}$, $\delta^3 t$ and $\delta\log(t+1)$ are sufficiently small.

Therefore, the proof of Lemma 2.6 is completed. □

3 Some properties for the viscous shock waves and the estimates for wave interactions

In this section, we first give some fundamental facts about the viscous shock waves. Then, we deal with the wave interactions from the two different characteristic fields. We begin with the following lemma concerning some properties of the viscous shock waves.

Lemma 3.1. *Let the viscosity coefficient ϵ be equal to one in* (1.11) *and* (1.12), *and denote $\bar{U}_1$ and $\bar{U}_2$ by the viscous shock waves of* (1.11) *and* (1.12). *Then, the two viscous shock waves $\bar{U}_1$ and $\bar{U}_2$ defined in* (1.11) *and* (1.12), *respectively satisfy the following estimates*:

$$|(\bar{U}_1-U_-)(x,t)|\leqslant C\delta_1\mathrm{e}^{-\delta_1|x-s_1t|/C},\quad x<s_1t,\quad t\geqslant 0, \tag{3.1}$$

$$|(\bar{U}_1-U_m)(x,t)|\leqslant C\delta_1\mathrm{e}^{-\delta_1|x-s_1t|/C},\quad x>s_1t,\quad t\geqslant 0, \tag{3.2}$$

$$|(\bar{U}_2-U_m)(x,t)|\leqslant C\delta_2\mathrm{e}^{-\delta_2|x-s_2t|/C},\quad x<s_2t,\quad t\geqslant 0, \tag{3.3}$$

$$|(\bar{U}_2-U_+)(x,t)|\leqslant C\delta_2\mathrm{e}^{-\delta_2|x-s_2t|/C},\quad x>s_2t,\quad t\geqslant 0, \tag{3.4}$$

$$U_{i,x}(x,t)<0,\quad |U_{i,x}|\leqslant C\delta_i^2\mathrm{e}^{-\delta_i|x-s_it|/C},\quad x\in\mathbb{R},\quad t\geqslant 0, \tag{3.5}$$

$$\|(\bar{U}_{\alpha_1,\alpha_2}-\bar{U}_{\mathrm{TW}})(\cdot,t)\|\leqslant C[\delta^{1/2}+\delta^2t^{1/2}+\delta^{7/2}t], \tag{3.6}$$

$$\int_{-\infty}^0\left|\int_{-\infty}^x(\bar{U}_{1,\alpha_1}-\bar{U}^1_{\mathrm{TW}})(y,t)dy\right|^2dx\leqslant C\sum_{a-2b\geqslant 0}\delta^a(t+1)^b, \tag{3.7}$$

$$\int_0^{\infty}\left|\int_x^{\infty}(\bar{U}_{2,\alpha_2}-\bar{U}^2_{\mathrm{TW}})(y,t)dy\right|^2dx\leqslant C\sum_{a-2b\geqslant 0}\delta^a(t+1)^b, \tag{3.8}$$

$$\int_0^{\infty}\left|\int_x^{\infty}(\bar{U}_{1,\alpha_1}-U_m)(y,t)dy\right|^2dx\leqslant C[\delta^{-1}\mathrm{e}^{-\delta t/C}+\delta^2], \tag{3.9}$$

$$\int_{-\infty}^0\left|\int_{-\infty}^x(\bar{U}_{2,\alpha_2}-U_m)(y,t)dy\right|^2dx\leqslant C[\delta^{-1}\mathrm{e}^{-\delta t/C}+\delta^2], \tag{3.10}$$

where $\bar{U}_{i,\alpha_i}(x,t)=\bar{U}_i(x-s_it+\alpha_i)$ $(i=1,2)$ and $\bar{U}_{\alpha_1,\alpha_2}=\bar{U}_{1,\alpha_1}+\bar{U}_{2,\alpha_2}-U_m$.

Proof. Since the proofs of (3.1)–(3.5) are fundamental, we will focus on the proofs of (3.6)–(3.10). To begin with, by noting that (1.9), (1.10) and (2.12), then we can follow the arguments in [9] step by step to obtain

$$|\bar{U}_1(x)-\bar{U}^1_{\mathrm{TW}}(x)|\leqslant\begin{cases}C\delta^2\mathrm{e}^{-\delta|x|/C}, & x\leqslant 0,\\ C(\delta^3+\delta^2\mathrm{e}^{-\delta|x|/C}), & x\geqslant 0,\end{cases} \tag{3.11}$$

and

$$|\bar{U}_2(x)-\bar{U}^2_{\mathrm{TW}}(x)|\leqslant\begin{cases}C(\delta^3+\delta^2\mathrm{e}^{-\delta|x|/C}), & x\leqslant 0,\\ C\delta^2\mathrm{e}^{-\delta|x|/C}, & x\geqslant 0\end{cases} \tag{3.12}$$

(see the proof of [9, (5.17)]). By virtual of (1.3), (1.9), (1.13), (1.14) and (3.1)–(3.4), we have the following critical estimates on the phase shifts:

$$|\alpha_i|=O(\delta^{-1}),\quad i=1,2. \tag{3.13}$$

To prove (3.6), one has

$$\begin{aligned}\|(\bar{U}_{\alpha_1,\alpha_2}-\bar{U}_{\mathrm{TW}})(\cdot,t)\|^2&=\|(\bar{U}_{\alpha_1,\alpha_2}-\bar{U}_{\mathrm{TW}})(\cdot,t)\|^2_{L^2(\mathbb{R}^-)}+\|(\bar{U}_{\alpha_1,\alpha_2}-\bar{U}_{\mathrm{TW}})(\cdot,t)\|^2_{L^2(\mathbb{R}^+)}\\&=J_1+J_2.\end{aligned}\tag{3.14}$$

Now, we treat the terms J_1 and J_2 separately. Using the Cauchy inequality, we have

$$\begin{aligned}J_1&=\int_{-\infty}^0|\bar{U}_{1,\alpha_1}(x,t)+\bar{U}_{2,\alpha_2}(x,t)-U_m-\bar{U}^1_{\mathrm{TW}}(x-\tilde{s}_1t)-\bar{U}^2_{\mathrm{TW}}(x-\tilde{s}_2t)+\tilde{U}_m|^2dx\\&\leqslant 3\bigg[\int_{-\infty}^0|\bar{U}_{1,\alpha_1}(x,t)-\bar{U}^1_{\mathrm{TW}}(x-\tilde{s}_1t)|^2dx+\int_{-\infty}^0|\bar{U}_{2,\alpha_2}(x,t)-U_m|^2dx\\&\quad+\int_{-\infty}^0|\bar{U}^2_{\mathrm{TW}}(x-\tilde{s}_2t)-\tilde{U}_m|^2dx\bigg]\\&=J_1^1+J_1^2+J_1^3.\end{aligned}\tag{3.15}$$

By virtue of (3.5), (3.11) and (3.13), we get

$$\begin{aligned}J_1^1&\leqslant C\bigg[\int_{-\infty}^0|(\bar{U}_1-\bar{U}^1_{\mathrm{TW}})(x+\alpha_1-s_1t)|^2dx+\int_{-\infty}^0|\bar{U}^1_{\mathrm{TW}}(x+\alpha_1-s_1t)-\bar{U}^1_{\mathrm{TW}}(x-\tilde{s}_1t)|^2dx\bigg]\\&\leqslant C\bigg[\int_{-\infty}^{|\alpha_1|-s_1t}|(\bar{U}_1-\bar{U}^1_{\mathrm{TW}})(x)|^2dx+(1+|s_1-\tilde{s}_1|^2t^2)\int_R\bigg|\frac{\partial\bar{U}^1_{\mathrm{TW}}}{\partial x}\bigg|^2dx\bigg]\\&\leqslant C[\delta^3+\delta^4t+\delta^7t^2],\end{aligned}\tag{3.16}$$

where we have used the fact that

$$s_1=\frac{\lambda_1(U_-)+\lambda_1(U_m)}{2}+O(\delta^2)=\frac{\lambda_1(U_-)+\lambda_1(\tilde{U}_m)}{2}+O(\delta^2)=\tilde{s}_1+O(\delta^2).$$

Applying the bound (3.5) and (3.13), we obtain

$$J_1^2=\int_{-\infty}^{\alpha_2-s_2t}|\bar{U}_2(x)-U_m|^2dx\leqslant C\bigg[\int_{-\infty}^0|\bar{U}_2(x)-U_m|^2dx+\int_0^{|\alpha_2|}|\bar{U}_2(x)-U_m|^2dx\bigg]\leqslant C\delta.\tag{3.17}$$

From (2.27) and (2.29), we get

$$\begin{aligned}J_1^3&\leqslant C\tilde{\delta}_2^2\int_{-\infty}^0\frac{1}{(1+\mathrm{e}^{-\tilde{\delta}_2(x-\tilde{s}_2(t+1))/\beta})^2}dx\leqslant C\tilde{\delta}_2^2\int_{-\infty}^{-\tilde{s}_2(t+1)}\frac{1}{(1+\mathrm{e}^{-\tilde{\delta}_2(x-\tilde{s}_2(t+1))/\beta})^2}dx\\&\leqslant C\tilde{\delta}_2^2\int_{-\infty}^0\mathrm{e}^{\tilde{\delta}_2x/C}dx\leqslant C\delta.\end{aligned}\tag{3.18}$$

Inserting the bounds (3.16)–(3.18) into (3.15), we obtain

$$J_1\leqslant C[\delta+\delta^4t+\delta^7t^2].\tag{3.19}$$

To control J_2, we triangulate as follows

$$\bar{U}_{\alpha_1,\alpha_2}-\bar{U}_{\mathrm{TW}}=(\bar{U}_{2,\alpha_2}-\bar{U}^2_{\mathrm{TW}})+(\bar{U}_{1,\alpha_1}-U_m)+(\tilde{U}_m-\bar{U}^1_{\mathrm{TW}}).\tag{3.20}$$

Then, using the similar arguments just as before, we can obtain

$$J_2\leqslant C[\delta+\delta^4t+\delta^7t^2].\tag{3.21}$$

Therefore, (3.6) follows from (3.19) and (3.21) immediately.

Now, we turn to the proof of (3.7). By virtue of (2.27), (2.29) and (3.11), we have

$$\int_{-\infty}^0\bigg|\int_{-\infty}^x(\bar{U}_{1,\alpha_1}-\bar{U}^1_{\mathrm{TW}})(y,t)dy\bigg|^2dx$$

$$
\begin{aligned}
&\leqslant C\int_{-\infty}^{\alpha_1-s_1t}\left(\int_{-\infty}^{x}|(\bar U_1-\bar U^1_{\rm TW})(y)|dy\right)^2dx\\
&\quad+C\int_{-\infty}^{0}\left(\int_{-\infty}^{x}|\bar U^1_{\rm TW}(y+\alpha_1-s_1t)-\bar U^1_{\rm TW}(y-\tilde s_1t)|dy\right)^2dx.
\end{aligned}\tag{3.22}
$$

The first integral on the right-side hand of (3.22) can be bounded as follows:

$$
\begin{aligned}
&C\int_{-\infty}^{\alpha_1-s_1t}\left(\int_{-\infty}^{x}|(\bar U_1-\bar U^1_{\rm TW})(y)|dy\right)^2dx\\
&\leqslant C\int_{-\infty}^{0}\left(\int_{-\infty}^{x}|(\bar U_1-\bar U^1_{\rm TW})(y)|dy\right)^2dx+C\int_{0}^{|\alpha_1|-s_1t}\left(\int_{-\infty}^{0}|(\bar U_1-\bar U^1_{\rm TW})(y)|dy\right)^2dx\\
&\quad+C\int_{0}^{|\alpha_1|-s_1t}\left(\int_{0}^{x}|(\bar U_1-\bar U^1_{\rm TW})(y)|dy\right)^2dx\\
&\leqslant C[\delta+\delta^2(t+1)+\delta^6(t+1)^3].
\end{aligned}
$$

By noting (2.27), (2.29) and (3.13), it is easy to see that the second integral can be bounded by

$$
C(1+|s_1-\tilde s_1|^2t^2)\int_{-\infty}^{O(1)(t+1)}\left(\int_{-\infty}^{x}|(\bar U^1_{\rm TW})'(z)|dz\right)^2dx.
$$

Using the fact that $|(\bar U^1_{\rm TW})'(z)|\leqslant C\delta^2{\rm e}^{-\delta|z|/C}$ (see (2.27) and (2.29)) and that $|s_1-\tilde s_1|\leqslant C\delta^2$, we find that the second term on the right-side hand of (3.21) can be bounded by $C[\delta^2(t+1)+\delta^6(t+1)^3]$. These estimates prove (3.7). The proof of (3.8) is similar.

Finally, we prove the bounds (3.9) and (3.10). It is sufficient to prove (3.9), since the proof of (3.10) is similar. By noting that the shock wave $s_1<0$, (1.9), (1.10), (3.2), (3.5) and (3.13), we have

$$
\begin{aligned}
&\int_0^\infty\left|\int_x^\infty(\bar U_{1,\alpha_1}-U_m)(y,t)dy\right|^2dx\\
&\leqslant C\int_0^\infty\left|\int_x^\infty(\bar U_1-U_m)(y,t)dy\right|^2dx+C\int_0^\infty\left|\int_x^\infty(\bar U_{1,\alpha_1}-\bar U_1)(y,t)dy\right|^2dx\\
&\leqslant C[\delta^{-1}{\rm e}^{-\delta t/C}+\delta^2].
\end{aligned}\tag{3.23}
$$

This proves the estimate (3.9). Therefore, the proof of Lemma 3.1 is completed. □

To deal with the wave interactions from the two different characteristic fields, we divide $R\times(0,t)$ into two parts as $R\times(0,t)=\Omega^-\cup\Omega^+$, where

$$
\Omega^-=\left\{(x,t)\,\middle|\,x\leqslant\frac{s_1+s_2}{2}t\right\}\quad\text{and}\quad\Omega^+=\left\{(x,t)\,\middle|\,x>\frac{s_1+s_2}{2}t\right\}.
$$

Then, we have the following lemma concerning the wave interactions estimates:

Lemma 3.2. *Let the two viscous shock waves $\bar U_{1,\alpha_1}$ and $\bar U_{2,\alpha_2}$ be as defined above. Then*

$$|(\bar U_{1,\alpha_1}-U_m)(x,t)|=O(1)\delta_1{\rm e}^{-\delta_1(|x|+t)/C},\quad in\ \Omega^+,\tag{3.24}$$

$$|(\bar U_{2,\alpha_2}-U_m)(x,t)|=O(1)\delta_2{\rm e}^{-\delta_2(|x|+t)/C},\quad in\ \Omega^-,\tag{3.25}$$

$$|(\bar U_{1,\alpha_1}-U_m)(x,t)||(\bar U_{2,\alpha_2})_x(x,t)|=O(1)\delta_2^2\delta_1({\rm e}^{-\delta_1(|x|+t)/C}+{\rm e}^{-\delta_2(|x|+t)/C}),\tag{3.26}$$

$$|(\bar U_{2,\alpha_2}-U_m)(x,t)||(\bar U_{1,\alpha_1})_x(x,t)|=O(1)\delta_1^2\delta_2({\rm e}^{-\delta_1(|x|+t)/C}+{\rm e}^{-\delta_2(|x|+t)/C}).\tag{3.27}$$

Proof. Since the inviscid system (1.6) is strictly hyperbolic, then $s_1<0<s_2$. When t is large, the two shock waves will decouple. With this, (3.24)–(3.27) can be proved easily.

Indeed, set $t_0=4\max_{i=1,2}\{|\alpha_i|\}/s_2$. When $t\leqslant t_0$, the proofs of the estimates (3.24)–(3.27) are obvious. When $t>t_0$, we have, in Ω^+,

$$
x+\alpha_1-s_1t>(s_1+s_2)t/2+\alpha_1-s_1t\geqslant(s_2-s_1)t/4>0,\tag{3.28}
$$

and by Lemma 3.1,

$$|(\bar{U}_{1,\alpha_1} - U_m)(x,t)| = O(1)\delta_1 \mathrm{e}^{-\delta_1|x+\alpha_1-s_1t|/C}. \tag{3.29}$$

It is clear that $|x + \alpha_1 - s_1t| \geqslant Ct$, by (3.28).

When $x \leqslant 0$, we have

$$|x + \alpha_1 - s_1t| = x + \alpha_1 - s_1t \geqslant x + (s_1 - s_2)t/4 - s_1t \geqslant x - 3x/2 = |x|/2. \tag{3.30}$$

When $x > 0$, if $s_1 + s_2 \leqslant 0$,

$$|x + \alpha_1 - s_1t| \geqslant x + \alpha_1 + s_2t \geqslant x + 3|\alpha_1| \geqslant x = |x|; \tag{3.31}$$

if $s_1 + s_2 > 0$,

$$|x + \alpha_1 - s_1t| \geqslant x - (s_1 + s_2)t/4 \geqslant x - x/2 = |x|/2. \tag{3.32}$$

Therefore, $|x + \alpha_1 - s_1t| \geqslant C|x|$. Now, we have proved the estimate (3.24). The other estimates in Lemma 3.2 can be treated similarly. So, we omit the details.

Therefore, the proof of Lemma 3.2 is completed. □

4 Intermediate-time estimate for $U - \bar{U}_{\alpha_1,\alpha_2}$

In this section, we use Lemma 3.1, together with the estimates of Lemma 2.6, to give L^2 bounds for $U - \bar{U}_{\alpha_1,\alpha_2}$ and its x-antiderivative at time $t \leqslant \delta^{-2-\vartheta}$. These bounds will serve to control the "initial data" for the energy estimates of Section 5, where we finally conclude the proof of Theorem 1.1.

Lemma 4.1. *Under the assumptions of Lemma* 2.6*, for* $t \leqslant \delta^{-2-\vartheta}$*, we have*

$$\|U - \bar{U}_{\alpha_1,\alpha_2}\|^2 + \|(v - \bar{v}_{\alpha_1,\alpha_2})_x\|^2 \leqslant C \sum_{a-2b\geqslant 1} \delta^a (t+1)^b, \tag{4.1}$$

and

$$\left\| \int_{-\infty}^{x} (U - \bar{U}_{\alpha_1,\alpha_2})(y,t)dy \right\|^2 \leqslant C\left[\sum_{a-2b\geqslant 0} \delta^a (t+1)^b + (\delta^{-1} + (t+1)^{1/2})\mathrm{e}^{-\delta^2 t/C} + \delta^2 (t+1)^{3/2} \mathrm{e}^{-t/C} \right]. \tag{4.2}$$

In addition, there is a positive number M*, depending only on* U_- *and* U_+*, such that, if*

$$M\delta^{-2}\log\delta^{-1} \leqslant t \leqslant \delta^{-2-\vartheta}$$

(*which is possible for small* δ)*, then at time* t*,*

$$\|U - \bar{U}_{\alpha_1,\alpha_2}\|^2 + \|(v - \bar{v}_{\alpha_1,\alpha_2})_x\|^2 \leqslant C\delta^{1-\vartheta B} \tag{4.3}$$

and

$$\left\| \int_{-\infty}^{x} (U - \bar{U}_{\alpha_1,\alpha_2})(y,t)dy \right\|^2 \leqslant C\delta^{-\vartheta B}, \tag{4.4}$$

where B *is a positive constant depending only on* U_-*.*

Proof. It is clear that the bounds (4.3) and (4.4) follow from (4.1) and (4.2), respectively. Therefore, it is sufficient to prove (4.1) and (4.2). To begin with, we triangulate as follows:

$$U - \bar{U}_{\alpha_1,\alpha_2} = (U - \bar{U}_R - \bar{\bar{U}}) + (\bar{U}_R - \bar{U}_{\mathrm{TW}}) + \bar{\bar{U}} + (\bar{U}_{\mathrm{TW}} - \bar{U}_{\alpha_1,\alpha_2}).$$

Bounds for the L^2-norms of these terms are given in (2.64), (2.49), (2.54) and (3.6), respectively. This proves the estimate (4.1). To prove (4.2), we treat the case $x \leqslant 0$ and $x \geqslant 0$ separately. First, for $x \leqslant 0$, we triangulate as follows,

$$U-\bar{U}_{\alpha_1,\alpha_2}=(U-\bar{U}_R-\bar{\bar{U}})+(\bar{U}_R^1-\bar{U}_{\mathrm{TW}}^1)+(\bar{U}_R^2-\tilde{U}_m)+\bar{\bar{U}}+(\bar{U}_{\mathrm{TW}}^1-\bar{U}_{1,\alpha_1})+(U_m-\bar{U}_{2,\alpha_2}).$$

Bounds for the antiderivatives of these terms are given in (2.63), (2.50), (2.37), (2.61), (3.7) and (3.10), respectively. This gives an estimate for $\int_{-\infty}^0|\int_{-\infty}^x(U-\bar{U}_{\alpha_1,\alpha_2})(y,t)dy|^2dx$. For $x \geqslant 0$, we triangulate differently,

$$U-\bar{U}_{\alpha_1,\alpha_2}=(U-\bar{U}_R-\bar{\bar{U}})+(\bar{U}_R^2-\bar{U}_{\mathrm{TW}}^2)+(\bar{U}_R^1-\tilde{U}_m)+\bar{\bar{U}}+(\bar{U}_{\mathrm{TW}}^2-\bar{U}_{2,\alpha_2})+(U_m-\bar{U}_{1,\alpha_1}).$$

The appropriate bounds are obtained in (2.63), (2.51), (2.36), (2.62), (3.8) and (3.9), respectively, thereby giving an estimate for $\int_0^\infty|\int_x^\infty(U-\bar{U}_{\alpha_1,\alpha_2})(y,t)dy|^2dx$. Combining the above, and noting that

$$\int_{-\infty}^x(U-\bar{U}_{\alpha_1,\alpha_2})(y,t)dy=-\int_x^\infty(U-\bar{U}_{\alpha_1,\alpha_2})(y,t)dy$$

by (1.11), (1.12), (1.14)–(1.16) and (2.1), we finally get (4.2). □

5 Proof of Theorem 1.1

In this section, we combine the results of the previous sections to complete the proof of Theorem 1.1.

Set

$$\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}=\int_{-\infty}^x(U-\bar{U}_{\alpha_1,\alpha_2})(y,t)dy\quad\text{and}\quad\begin{bmatrix}\phi\\ \psi\end{bmatrix}=U-\bar{U}_{\alpha_1,\alpha_2}. \tag{5.1}$$

First, we show the following a priori estimates:

Lemma 5.1. *Given $U_-=\left[\begin{smallmatrix}v_-\\ v_+\end{smallmatrix}\right]$, there is a small constant ε_0 depending only on U_- such that, if $\delta=|U_+-U_-|\leqslant\varepsilon_0$ and if a solution U to (1.3)–(2.1) exists for $t_0\leqslant t\leqslant t_1$ and satisfies*

$$\|\Psi\|_{L^\infty},\quad \|v-v_-\|_{L^\infty}\leqslant\varepsilon_0,$$

then

$$\begin{aligned}\sup_{t_0\leqslant t\leqslant t_1}\|(\Phi,\Psi)(t)\|^2&+\int_{t_0}^{t_1}\|(\Phi_x,\Psi_x)(t)\|^2dt+\int_{t_0}^{t_1}\int_R(|(U_{1,\alpha_1})_x|+|(U_{2,\alpha_2})_x|)\Psi^2dxdt\\ &\leqslant C\left[\|(\Phi,\Psi)(t_0)\|^2+\sup_{t_0\leqslant t\leqslant t_1}\|\phi(t)\|^2+\delta+\int_{t_0}^{t_1}\|\psi_x(t)\|^2dt\right]\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}\sup_{t_0\leqslant t\leqslant t_1}[\|(\phi,\psi)(t)\|^2+\|\phi_x(t)\|^2]&+\int_{t_0}^{t_1}\|(\phi_x,\psi_x)(t)\|^2dt\\ &\leqslant C\left[\|(\phi,\psi)(t_0)\|^2+\|\phi_x(t_0)\|^2+\delta+\delta\int_{t_0}^{t_1}\|\phi(t)\|^2dt\right].\end{aligned}\tag{5.3}$$

Proof. First, by virtue of (5.1), we can get

$$\begin{cases}\Phi_t-\Psi_x=0,\\ \Psi_t+p(\bar{v}_{\alpha_1,\alpha_2}+\Phi_x)-p(\bar{v}_{\alpha_1,\alpha_2})\\ \qquad=\Psi_{xx}/v+(1/v-1/V_{1,\alpha_1})(U_{1,\alpha_1})_x+(1/v-1/V_{2,\alpha_2})(U_{2,\alpha_2})_x\\ \qquad\quad-[p(\bar{v}_{\alpha_1,\alpha_2})-p(V_{1,\alpha_1})-p(V_{2,\alpha_2})+p(v_m)],\end{cases}\tag{5.4}$$

and

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + (p(\bar{v}_{\alpha_1,\alpha_2} + \phi) - p(\bar{v}_{\alpha_1,\alpha_2}))_x \\ \qquad = [\psi_x/v + (1/v - 1/V_{1,\alpha_1})(U_{1,\alpha_1})_x + (1/v - 1/V_{2,\alpha_2})(U_{2,\alpha_2})_x \\ \qquad\quad -(p(\bar{v}_{\alpha_1,\alpha_2}) - p(V_{1,\alpha_1}) - p(V_{2,\alpha_2}) + p(v_m))]_x, \end{cases} \tag{5.5}$$

where, $V_{i,\alpha_i} = V_i(x + \alpha_i - s_i t)$, etc.

Linearizing the second equation in (5.4), we have

$$\begin{aligned} \Psi_t + p'(\bar{v}_{\alpha_1,\alpha_2})\Phi_x = {} & \Psi_{xx}/\bar{v}_{\alpha_1,\alpha_2} + (1/v - 1/\bar{v}_{\alpha_1,\alpha_2})\Psi_{xx} \\ & + (1/v - 1/V_{1,\alpha_1})(U_{1,\alpha_1})_x + (1/v - 1/V_{2,\alpha_2})(U_{2,\alpha_2})_x \\ & - [p(\bar{v}_{\alpha_1,\alpha_2}) - p(V_{1,\alpha_1}) - p(V_{2,\alpha_2}) + p(v_m)] + O(\Phi_x^2). \end{aligned} \tag{5.6}$$

We multiply the first equation in (5.4) and (5.6) by Φ and $-\Psi/p'(\bar{v}_{\alpha_1,\alpha_2})$, respectively, and add and integrate. Noting that $(1/p'(\bar{v}_{\alpha_1,\alpha_2}))_t = p''(|(U_{1,\alpha_1})_x| + |(U_{2,\alpha_2})_x|)/(p')^2$, by (3.5) and (1.2), we get

$$\begin{aligned} & \sup_{t_0 \leqslant t \leqslant t_1} \|(\Phi,\Psi)(t)\|^2 + \int_{t_0}^{t_1}\int_R (|(U_{1,\alpha_1})_x| + |(U_{2,\alpha_2})_x|)\Psi^2 dxdt + \int_{t_0}^{t_1} \|\Psi_x\|^2 dt \\ & \leqslant C\|(\Phi,\Psi)(t_0)\|^2 + C\int_{t_0}^{t_1}\int_R \{|(U_{1,\alpha_1})_x| + |(U_{2,\alpha_2})_x|)|\Psi\Psi_{xx}| + |\Phi_x\Psi\Psi_{xx}| \\ & \quad +(|\Phi_x| + |V_{2,\alpha_2} - v_m|)|(U_{1,\alpha_1})_x||\Psi| + (|\Phi_x| + |V_{1,\alpha_1} - v_m|)|(U_{2,\alpha_2})_x||\Psi| \\ & \quad +|p(\bar{v}_{\alpha_1,\alpha_2}) - p(V_{1,\alpha_1}) - p(V_{2,\alpha_2}) + p(v_m)||\Psi| + |\Psi|\Phi_x^2\} dxdt \\ & = C\|(\Phi,\Psi)(t_0)\|^2 + \sum_{i=1}^{6} K_i. \end{aligned} \tag{5.7}$$

Using the Young's inequality, we have

$$K_1 \leqslant \frac{1}{4}\int_{t_0}^{t_1}\int_R (|(U_{1,\alpha_1})_x| + |(U_{2,\alpha_2})_x|)\Psi^2 dxdt + C\int_{t_0}^{t_1}\int_R (|(U_{1,\alpha_1})_x| + |(U_{2,\alpha_2})_x|)\Psi_{xx}^2 dxdt, \tag{5.8}$$

and

$$K_2 \leqslant C\|\Psi\|_{L^\infty}\int_{t_0}^{t_1}(\|\Phi_x\|^2 + \|\Psi_{xx}\|^2)dt. \tag{5.9}$$

By virtue of Young's inequality and Lemma 3.2, we obtain

$$\begin{aligned} K_3 & \leqslant \frac{1}{4}\int_{t_0}^{t_1}\int_R |(U_{1,\alpha_1})_x||\Psi|^2 dxdt + C\int_{t_0}^{t_1}\int_R |(U_{1,\alpha_1})_x|\Phi_x^2 \\ & \quad + C\int_{t_0}^{t_1}\int_R |(U_{1,\alpha_1})_x||V_{2,\alpha_2} - v_m|^2 dxdt \\ & \leqslant \frac{1}{4}\int_{t_0}^{t_1}\int_R |(U_{1,\alpha_1})_x||\Psi|^2 dxdt + C\int_{t_0}^{t_1}\int_R |(U_{1,\alpha_1})_x|\Phi_x^2 \\ & \quad + C\delta_1^2\delta_2^2\int_{t_0}^{t_1}\int_R (\mathrm{e}^{-\delta_1(|x|+t)/C} + \mathrm{e}^{-\delta_2(|x|+t)/C})dxdt \\ & \leqslant \frac{1}{4}\int_{t_0}^{t_1}\int_R |(U_{1,\alpha_1})_x||\Psi|^2 dxdt + C\int_{t_0}^{t_1}\int_R |(U_{1,\alpha_1})_x|\Phi_x^2 + C\delta^2. \end{aligned} \tag{5.10}$$

Similarly, we have

$$K_4 \leqslant \frac{1}{4}\int_{t_0}^{t_1}\int_R |(U_{2,\alpha_2})_x||\Psi|^2 dxdt + C\int_{t_0}^{t_1}\int_R |(U_{2,\alpha_2})_x|\Phi_x^2 + C\delta^2. \tag{5.11}$$

Applying the mean value theorem, Lemma 3.2, (1.9), (1.10) and Young's inequality , we get

$$\begin{aligned}
K_5 &\leqslant C\int_{t_0}^{t_1}\int_R |V_{1,\alpha_1}-v_m||V_{2,\alpha_2}-v_m||\Psi|dxdt\\
&\leqslant C\delta_1\delta_2\int_{t_0}^{t_1}\int_R (\mathrm{e}^{-\delta_1(|x|+t)/C}+\mathrm{e}^{-\delta_2(|x|+t)/C})|\Psi|dxdt\\
&\leqslant C\bar{\delta}^2\int_{t_0}^{t_1}\int_R \mathrm{e}^{-\bar{\delta}(|x|+t)/C}|\Psi|dxdt\\
&\leqslant C\bar{\delta}^3\int_{t_0}^{t_1}\int_R \mathrm{e}^{-\bar{\delta}(|x|+t)/C}dxdt+\frac{\bar{\delta}}{2C}\int_{t_0}^{t_1}\mathrm{e}^{-\bar{\delta}t/C}\|\Psi(t)\|^2dt\\
&\leqslant C\delta+\frac{1}{2}\sup_{t_0\leqslant t\leqslant t_1}\|\Psi(t)\|^2dt. \qquad (5.12)
\end{aligned}$$

It is easy to see that

$$K_6\leqslant C\|\Psi\|_{L^\infty}\int_{t_0}^{t_1}\|\Phi_x\|^2dt. \qquad (5.13)$$

Combining the above relations (5.7)–(5.13), we obtain

$$\begin{aligned}
&\sup_{t_0\leqslant t\leqslant t_1}\|(\Phi,\Psi)(t)\|^2+\int_{t_0}^{t_1}\int_R(|(U_{1,\alpha_1})_x|+|(U_{2,\alpha_2})_x|)\Psi^2dxdt+\int_{t_0}^{t_1}\|\Psi_x(t)\|^2dt\\
&\quad\leqslant C\bigg[\|(\Phi,\Psi)(t_0)\|^2+\delta+(\|\Psi\|_{L^\infty}+|(U_{1,\alpha_1})_x|+|(U_{2,\alpha_2})_x|)\int_{t_0}^{t_1}(\|\Phi_x(t)\|^2+\|\Psi_{xx}(t)\|^2)dt\bigg]. \qquad (5.14)
\end{aligned}$$

Next, we estimate the term $\int_{t_0}^{t_1}\|\Phi_x\|^2dt$. We differentiate the first equation in (5.4) with respect to x, multiply by $-\Psi$, and add to $-\Phi_x$ times (5.6). Integrating, we then get

$$\begin{aligned}
\int_{t_0}^{t_1}\|\Phi_x(t)\|^2dt &\leqslant C\bigg[\sup_{t_0\leqslant t\leqslant t_1}(\|(\Psi(t)\|^2+\|\Phi_x(t)\|^2)+\int_{t_0}^{t_1}\|\Psi_x(t)\|^2dt\bigg]\\
&\quad+C\int_{t_0}^{t_1}\int_R|\Phi_x|\{|\Psi_{xx}|+|v-\bar{v}_{\alpha_1,\alpha_2}||\Psi_{xx}|+|v-V_{1,\alpha_1}||(U_{1,\alpha_1})_x|\\
&\quad+|v-V_{2,\alpha_2}||(U_{2,\alpha_2})_x|+|p(\bar{v}_{\alpha_1,\alpha_2})-p(V_{1,\alpha_1})-p(V_{2,\alpha_2})+p(v_m)|\}dxdt. \qquad (5.15)
\end{aligned}$$

Using the similar arguments just to that as before, we can obtain

$$\int_{t_0}^{t_1}\|\Phi_x(t)\|^2dt\leqslant C\bigg[\sup_{t_0\leqslant t\leqslant t_1}(\|\Psi(t)\|^2+\|\Phi_x(t)\|^2)+\delta^2+\int_{t_0}^{t_1}(\|\Psi_x\|^2+\|\Psi_{xx}\|^2)dt\bigg]. \qquad (5.16)$$

Then, (5.2) follows from (5.14) and (5.16) immediately. Now, we turn to the proof of the estimate (5.3). Multiplying the first equation in (5.5) by $[p(\bar{v}_{\alpha_1,\alpha_2})-p(\bar{v}_{\alpha_1,\alpha_2}+\phi)]$ and the second equation in (5.5) by ψ and adding, we have

$$\begin{aligned}
&\bigg[\frac{\psi^2}{2}+(p(\bar{v}_{\alpha_1,\alpha_2})\phi-\int_{\bar{v}_{\alpha_1,\alpha_2}}^{\bar{v}_{\alpha_1,\alpha_2}-\phi}p(\tau)d\tau)\bigg]_t\\
&\quad=-[\psi(p(\bar{v}_{\alpha_1,\alpha_2}+\phi)-p(\bar{v}_{\alpha_1,\alpha_2}))]_x-[p(\bar{v}_{\alpha_1,\alpha_2}+\phi)-p(\bar{v}_{\alpha_1,\alpha_2})-p'(\bar{v}_{\alpha_1,\alpha_2})\phi](\bar{v}_{\alpha_1,\alpha_2})_t\\
&\qquad+\psi[\psi_x/v+(1/v-1/V_{1,\alpha_1})(U_{1,\alpha_1})_x+(1/v-1/V_{2,\alpha_2})(U_{2,\alpha_2})_x\\
&\qquad-(p(\bar{v}_{\alpha_1,\alpha_2})-p(V_{1,\alpha_1})-p(V_{2,\alpha_2})+p(v_m))]_x. \qquad (5.17)
\end{aligned}$$

Using the Taylor formula, we get

$$C^{-1}\phi^2\leqslant p(\bar{v}_{\alpha_1,\alpha_2})\phi-\int_{\bar{v}_{\alpha_1,\alpha_2}}^{\bar{v}_{\alpha_1,\alpha_2}-\phi}p(\tau)d\tau\leqslant C\phi^2, \qquad (5.18)$$

and

$$C^{-1}\phi^2 \leqslant p(\bar{v}_{\alpha_1,\alpha_2}+\phi)-p(\bar{v}_{\alpha_1,\alpha_2})-p'(\bar{v}_{\alpha_1,\alpha_2})\phi \leqslant C\phi^2. \tag{5.19}$$

Integrating (5.17) over $R\times(t_0,t_1)$ and using (5.18) and (5.19), we then obtain

$$\begin{aligned}\sup_{t_0\leqslant t\leqslant t_1}\|(\phi,\psi)(t)\|^2 \leqslant C\bigg[&\|(\phi,\psi)(t_0)\|^2+\int_{t_0}^{t_1}\int_R |(\bar{v}_{\alpha_1,\alpha_2})_t|\phi^2 dxdt\\ &+\int_{t_0}^{t_1}\int_R \psi[\psi_x/v+(1/v-1/V_{1,\alpha_1})(U_{1,\alpha_1})_x+(1/v-1/V_{2,\alpha_2})(U_{2,\alpha_2})_x\\ &-(p(\bar{v}_{\alpha_1,\alpha_2})-p(V_{1,\alpha_1})-p(V_{2,\alpha_2})+p(v_m))\bigg]_x dxdt\\ =C[&\|(\phi,\psi)(t_0)\|^2+L_1+L_2.\end{aligned} \tag{5.20}$$

By virtue of Lemma 3.1, we get

$$L_1\leqslant C\delta^2\int_{t_0}^{t_1}\|\phi(t)\|^2dt. \tag{5.21}$$

If we integrate the term L_2 by parts with respect to x, the boundary term will be

$$C\int_{t_0}^{t_1}\psi(0,\tau)\bigg(\bigg[p(v)-\frac{u_x}{v}\bigg]\bigg)(0,\tau)d\tau$$

(the bracket denotes jump), which is identically zero by [9, Theorem 1.3]. Therefore,

$$\begin{aligned}L_2\leqslant&-C\int_{t_0}^{t_1}\|\psi_x(t)\|^2dt+C\int_{t_0}^{t_1}\int_R[|\psi_x||(U_{1,\alpha_1})_x||v-V_{1,\alpha_1}|\\ &+|\psi_x||(U_{2,\alpha_2})_x||v-V_{2,\alpha_2}|+|\psi_x||p(\bar{v}_{\alpha_1,\alpha_2})-p(V_{1,\alpha_1})-p(V_{2,\alpha_2})+p(v_m)|]dxdt\\ =&-C\int_{t_0}^{t_1}\|\psi_x(t)\|^2dt+L_2^1+L_2^2+L_2^3.\end{aligned} \tag{5.22}$$

By virtue of triangle inequality, Cauchy-Schwarz inequality, Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned}L_2^1\leqslant& C\int_{t_0}^{t_1}\int_R|\psi_x||(U_{1,\alpha_1})_x|(|V_{2,\alpha_2}-v_m|+|\phi|)dxdt\\ \leqslant& C\int_{t_0}^{t_1}\int_R|(U_{1,\alpha_1})_x||\psi_x|^2dxdt+C\int_{t_0}^{t_1}\int_R|(U_{1,\alpha_1})_x||V_{2,\alpha_2}-v_m|^2dxdt\\ &+C\int_{t_0}^{t_1}\int_R|(U_{1,\alpha_1})_x|(\psi_x^2+\phi^2)dxdt\\ \leqslant& C\delta^2\int_{t_0}^{t_1}(\|\phi(t)\|^2+\|\psi_x(t)\|^2)dt+C\delta^2.\end{aligned} \tag{5.23}$$

Similarly, we have

$$L_2^2\leqslant C\delta^2\int_{t_0}^{t_1}(\|\phi(t)\|^2+\|\psi_x(t)\|^2)dt+C\delta^2. \tag{5.24}$$

Applying mean value theorem, Cauchy-Schwarz inequality, Lemma 3.2, (1.9) and (1.10), we get

$$\begin{aligned}L_2^3\leqslant& C\int_{t_0}^{t_1}\int_R|V_{1,\alpha_1}-v_m||V_{2,\alpha_2}-v_m||\psi_x|dxdt\\ \leqslant& C\bar{\delta}\int_{t_0}^{t_1}\|\psi_x(t)\|^2dt+C\bar{\delta}^3\int_{t_0}^{t_1}\int_R \mathrm{e}^{-\bar{\delta}(|x|+t)/C}dxdt\\ \leqslant& C\delta\int_{t_0}^{t_1}\|\psi_x(t)\|^2dt+C\delta.\end{aligned} \tag{5.25}$$

Inserting the estimates (5.23)–(5.25) into (5.22), we obtain

$$L_2 \leqslant -C\int_{t_0}^{t_1}\|\psi_x(t)\|^2dt + C\delta\int_{t_0}^{t_1}(\|\phi(t)\|^2+\|\psi_x(t)\|^2)dt + C\delta. \tag{5.26}$$

Combining the relations (5.20), (5.21) and (5.26), we then get

$$\sup_{t_0\leqslant t\leqslant t_1}\|(\phi,\psi)(t)\|^2 + \int_{t_0}^{t_1}\|\psi_x(t)\|^2dt \leqslant C\Big[\|(\phi,\psi)(t_0)\|^2 + \delta\int_{t_0}^{t_1}\|\phi(t)\|^2dt + \delta\Big]. \tag{5.27}$$

Finally, we obtain a piecewise L^2 bound for the first order derivative of the variable ϕ to compete the proof of (5.3). Setting $H(x,t) = \log v - \log \bar{v}_{\alpha_1,\alpha_2}$ and using (1.11), (1.12) and (2.1), we can get

$$\begin{aligned}\psi_t + (p(v)-p(\bar{v}_{\alpha_1,\alpha_2}))_x = H_{xt} &+ [(1/\bar{v}_{\alpha_1,\alpha_2} - 1/V_{1,\alpha_1})(U_{1,\alpha_1})_x + (1/\bar{v}_{\alpha_1,\alpha_2} - 1/V_{2,\alpha_2})(U_{2,\alpha_2})_x \\ &- (p(\bar{v}_{\alpha_1,\alpha_2}) - p(V_{1,\alpha_1}) - p(V_{2,\alpha_2}) + p(v_m))]_x.\end{aligned} \tag{5.28}$$

Multiplying (5.28) by $-H_x$ and integrating, we obtain

$$\begin{aligned}&\sup_{t_0\leqslant t\leqslant t_1}\|H_x(t)\#^2 + \int_{t_0}^{t_1}\|H_x(t)\#^2dt \\ &\leqslant C\bigg[\|H_x(t_0)\#^2 + \int_{t_0}^{t_1}\fint|\phi||(\bar{v}_{\alpha_1,\alpha_2})_x||H_x|dxdt + \int_{t_0}^{t_1}\fint\psi_tH_xdxdt \\ &\quad + \int_{t_0}^{t_1}\fint|H_x|\{[(1/\bar{v}_{\alpha_1,\alpha_2} - 1/V_{1,\alpha_1})(U_{1,\alpha_1})_x + (1/\bar{v}_{\alpha_1,\alpha_2} - 1/V_{2,\alpha_2})(U_{2,\alpha_2})_x \\ &\quad + (p(\bar{v}_{\alpha_1,\alpha_2}) - p(V_{1,\alpha_1}) - p(V_{2,\alpha_2}) + p(v_m))]_x\}dxdt\bigg] \\ &= \sum_{i=1}^{6}M_i.\end{aligned} \tag{5.29}$$

By the definition of H and (3.5), we have

$$M_1 \leqslant C[\|\phi_x(t_0)\#^2 + \|(\bar{v}_{\alpha_1,\alpha_2})_x(t_0)\#^2] \leqslant C[\|\phi_x(t_0)\#^2 + \delta^3]. \tag{5.30}$$

Applying Cauchy-Schwarz inequality and (3.5), we get

$$M_2 \leqslant C\delta^2\int_{t_0}^{t_1}(\|H_x(t)\#^2 + \|\phi(t)\|^2)dt. \tag{5.31}$$

Noting

$$\psi_tH_x = (\psi H_x)_t - (\psi H_t)_x + H_t\psi_x = (\psi H_x)_t - (\psi H_t)_x + \psi_x^2/v + (1/v - 1/\bar{v}_{\alpha_1,\alpha_2})(\bar{u}_{\alpha_1,\alpha_2})_x\psi_x,$$

and [9, Theorem 1.3], we obtain

$$\begin{aligned}M_3 &\leqslant \sup_{t_0\leqslant t\leqslant t_1}\left(\frac{1}{4}\|H_x(t)\#^2 + C\|\psi(t)\|^2\right) + C\int_{t_0}^{t_1}\|\psi_x(t)\#^2dt \\ &\quad + C\int_{t_0}^{t_1}\fint|(\bar{u}_{\alpha_1,\alpha_2})_x||\phi||\psi_x|dxdt + C\int_{t_0}^{t_1}|[\psi H_t](0,t)|dt \\ &\leqslant \sup_{t_0\leqslant t\leqslant t_1}\left(\frac{1}{4}\|H_x(t)\#^2 + C\|\psi(t)\|^2\right) + C\int_{t_0}^{t_1}\|\psi_x(t)\#^2dt \\ &\quad + C\delta + C\delta^2\int_{t_0}^{t_1}\|\phi(t)\|^2dt.\end{aligned} \tag{5.32}$$

Using Cauchy-Schwarz inequality, Lemmas 3.1 and 3.2, we have

$$M_4 + M_5 \leqslant C\delta^2 + C\delta^2 \int_{t_0}^{t_1} \|H_x(t)\|^2 dt. \tag{5.33}$$

By using mean value theorem, Cauchy-Schwarz inequality and Lemma 3.2, we obtain

$$\begin{aligned} M_6 &\leqslant C \int_{t_0}^{t_1} \!\!\!\!\!\!\!\!\!\!\; \oint [|V_{1,\alpha_1} - v_m||(V_{2,\alpha_2})_x| + |V_{2,\alpha_2} - v_m||(V_{1,\alpha_1})_x|]|H_x| dxdt \\ &\leqslant C\delta^2 + C\delta^2 \int_{t_0}^{t_1} \|H_x(t)\|^2 dt. \end{aligned} \tag{5.34}$$

From (5.29)–(5.34), we get

$$\begin{aligned} &\sup_{t_0 \leqslant t \leqslant t_1} \|H_x(t)\|^2 + \int_{t_0}^{t_1} \|H_x(t)\|^2 dt \\ &\quad \leqslant C\bigg[\sup_{t_0 \leqslant t \leqslant t_1} \|\psi(t)\|^2 + \|\phi_x(t_0)\|^2 + \delta + \delta^2 \int_{t_0}^{t_1} \|\phi(t)\|^2 dt + \int_{t_0}^{t_1} \|\psi_x(t)\|^2 dt\bigg]. \end{aligned} \tag{5.35}$$

By the definition of H and (5.35), we have

$$\begin{aligned} &\sup_{t_0 \leqslant t \leqslant t_1} \|\phi_x(t)\|^2 + \int_{t_0}^{t_1} \|\phi_x(t)\|^2 dt \\ &\quad \leqslant C\bigg[\sup_{t_0 \leqslant t \leqslant t_1} \|(\phi,\psi)(t)\|^2 + \|\phi_x(t_0)\|^2 + \delta + \delta^2 \int_{t_0}^{t_1} \|\phi(t)\|^2 dt + \int_{t_0}^{t_1} \|\psi_x(t)\|^2 dt\bigg]. \end{aligned} \tag{5.36}$$

Then, (5.3) follows from (5.27) and (5.36) immediately, and the proof of Lemma 5.1 is completed. □

In the following lemma, we apply the local existence result in [9], the intermediate-time result, Lemma 2.6 and the a priori estimates above to obtain global existence. It should be mentioned that the local and global well-posedness of the system (1.1) or the corresponding non-isentropic system with discontinuous initial data have been systematically studied by Hoff, etc., see [9–15].

Lemma 5.2. *Given $U_- = \left[\begin{smallmatrix} lv_- \\ v_+ \end{smallmatrix}\right]$, there is a small constant ε_0 depending only on U_- such that, if $\delta = |U_+ - U_-| \leqslant \varepsilon_0$, then the Cauchy problem* (1.3)–(2.1) *has a unique global solution U satisfying*

$$\sup_{t \leqslant \tau} \|(\Phi,\Psi)(\tau)\|^2 + \int_t^\infty \|(\phi,\psi)(\tau)\|^2 d\tau \leqslant \begin{cases} C\delta^{-\vartheta B}[1 + \delta t + \delta^{-1} \mathrm{e}^{(-\delta^2 t)/C}], & t \leqslant t_0, \\ C\delta^{-\vartheta B}, & t \geqslant t_0, \end{cases} \tag{5.37}$$

$$\sup_{0 \leqslant \tau} \big[\|(\phi,\psi)(\tau)\|^2 + \|\phi_x(\tau)\|^2\big] + \int_0^\infty \|(\phi_x,\psi_x)\|^2 d\tau \leqslant C\delta^{1-\vartheta B}, \tag{5.38}$$

and

$$\begin{aligned} &\sup_{0<\tau} [g(\tau)\|u_x(\cdot,\tau)\|^2 + g(\tau)^2(\|u_\tau(\cdot,\tau)\|^2 + \|(u_x/v)_x(\cdot,\tau)\|^2)] \\ &\quad + \int_0^\infty [g(\tau)^2(\|u_\tau(\cdot,\tau)\|^2 + \|(u_x/v)_x(\cdot,\tau)\|^2) + g(\tau)^2\|u_{x\tau}(\cdot,\tau)\|^2] \leqslant C, \end{aligned} \tag{5.39}$$

where $t_0 = M\delta^{-2}\log\delta^{-1}$, M and B are positive constants depending only on U_-, and $g(\tau) = \min\{1,\tau\}$.

Proof. We take t_0 as indicated above, so that the estimates (4.3) and (4.4) hold. Adding a small multiple of (5.3) to δ times (5.2), we see that, for $t \geqslant t_0$,

$$\sup_{t_0 \leqslant t \leqslant t_1} \|(\Phi,\Psi)(t)\|^2 + \int_{t_0}^{t_1} \|(\Phi_x,\Psi_x)\|^2 dt + \int_{t_0}^{t_1} \int_R (|(U_{1,\alpha_1})_x| + |(U_{2,\alpha_2})_x|)\Psi^2 dxdt \leqslant C\delta^{-\vartheta B}, \tag{5.40}$$

and

$$\sup_{t_0\leqslant t\leqslant t_1}[\|(\phi,\psi)(t)\|^2+\|\phi_x(t)\|\!\!|^2]+\int_{t_0}^{t_1}\|(\phi_x,\psi_x)\|\!\!|^2dt\leqslant C\delta^{1-\vartheta B}. \tag{5.41}$$

These hold as a priori bounds, i.e., provided that $v-v_-$ and $\|\Psi\|_{L^\infty}$ remain small. On the other hand, (5.40) and (5.41) imply that $\|\Psi\|_{L^\infty}^4\leqslant C\|\Psi\|^2\|\psi\|^2\leqslant C\delta^{1-2\vartheta B}$ and $\|\phi\|_{L^\infty}^4\leqslant\|\phi\|^2\|\phi_x\|\!\!|^2\leqslant C\delta^{2-2\vartheta B}$, both of which are arbitrarily small, provided that δ is sufficiently small. These observations, together with the local existence result, [9, Theorem 1.3], and the intermediate-time result, Lemma 2.6, prove the global existence of U, and show that (5.40) and (5.41) hold for all $t\geqslant t_0$. (5.37) then follows from (5.40) and (4.2); (5.38) follows from (5.41), (4.1) and the triangulation

$$\begin{aligned}\int_0^{t_0}\|(\phi_x,\psi_x)\|\!\!|^2d\tau&\leqslant C\int_0^{t_0}\!\!\!\!\!\!-\!\!\!\!\!\!\int[|\Delta U_x|^2+|(\bar U_R)_x|^2+|\bar{\bar U}_x|^2+|(\bar U_{\alpha_1,\alpha_2})_x|^2]dxd\tau\\&\leqslant C\sum_{a-2b\geqslant1}\delta^a(t_0+1)^b\leqslant C\delta^{1-\vartheta B}\end{aligned}$$

by (2.64), (2.31), (2.56) and (3.5). Finally, (5.39) is a consequence of [9, Theorem 1.3] and the bounds (5.37) and (5.38). □

Proof of Theorem 1.1. First, it is easy to see that, if U^ϵ and U are the solutions to (1.1)–(1.3) and (1.3)–(2.1), respectively, then

$$U^\epsilon(x,t)=U\left(\frac{x}{\epsilon},\frac{t}{\epsilon}\right). \tag{5.42}$$

The global existence of U^ϵ, its regularity and the information (1.17) concerning the jump discontinuities in U^ϵ then follow directly from [9, Theorem 1.3] and Lemma 5.2. To prove Theorem 1.1, it is sufficient to prove the convergence results (1.18) and (1.19). Setting $G(t)=\|(U-\bar U_{\alpha_1,\alpha_2})(\cdot,t)\|^2$ and using (5.37)–(5.39) and (1.1), we have $\int_{t_0}^\infty\{G(t)+|\frac{d}{dt}G(t)|\}dt<\infty$. This yields $\lim_{t\to\infty}\|(U-\bar U_{\alpha_1,\alpha_2})(\cdot,t)\|^2=0$, from which and Sobolev's inequality it follows

$$\lim_{t\to\infty}\sup_{x\neq0}|U(x,t)-\bar U_{\alpha_1,\alpha_2}(x,t)|=0.$$

This together with the estimate (1.17) gives

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^1}|U(x,t)-\bar U_{\alpha_1,\alpha_2}(x,t)|=0. \tag{5.43}$$

Noting (1.15) and the definitions of $\bar U_1$, $\bar U_2$ and $\bar U_{\alpha_1,\alpha_2}$ (see Lemma 3.1), we have

$$\begin{aligned}\bar U^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon}(x,t)&=\bar U_1^\epsilon(x-s_1t+\alpha_1\epsilon)+\bar U_2^\epsilon(x-s_2t+\alpha_2\epsilon)-U_m\\&=\bar U_1\left(\frac{x-s_1t+\alpha_1\epsilon}{\epsilon}\right)+\bar U_2\left(\frac{x-s_2t+\alpha_2\epsilon}{\epsilon}\right)-U_m\\&=\bar U_1\left(\frac{x-s_1t}{\epsilon}+\alpha_1\right)+\bar U_2^\epsilon\left(\frac{x-s_2t}{\epsilon}+\alpha_2\right)-U_m\\&=\bar U_{\alpha_1,\alpha_2}\left(\frac{x}{\epsilon},\frac{t}{\epsilon}\right).\end{aligned} \tag{5.44}$$

Then, (1.19) follows from (5.42)–(5.44) immediately. By using (6.61) and (6.64), one can write

$$\begin{aligned}U^\epsilon(x,t)-U^0(x,t)&=U^\epsilon(x,t)-\bar U^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon}(x,t)+\bar U^\epsilon_{\alpha_1^\epsilon,\alpha_2^\epsilon}(x,t)-U^0(x,t)\\&=\left(U\left(\frac{x}{\epsilon},\frac{t}{\epsilon}\right)-\bar U_{\alpha_1,\alpha_2}\left(\frac{x}{\epsilon},\frac{t}{\epsilon}\right)\right)+\left(\bar U_{\alpha_1,\alpha_2}\left(\frac{x}{\epsilon},\frac{t}{\epsilon}\right)-U^0(x,t)\right).\end{aligned} \tag{5.45}$$

From (5.43), it is easy to see that

$$\lim_{\epsilon\to0}\sup_{|x-s_it|\geqslant h,i=1,2}\left|U\left(\frac{x}{\epsilon},\frac{t}{\epsilon}\right)-\bar U_{\alpha_1,\alpha_2}\left(\frac{x}{\epsilon},\frac{t}{\epsilon}\right)\right|=0. \tag{5.46}$$

By virtue of (1.20) and Lemma 3.1, we have

$$\lim_{\epsilon\to 0}\sup_{|x-s_it|\geqslant h, i=1,2}\left|\bar{U}_{\alpha_1,\alpha_2}\left(\frac{x}{\epsilon},\frac{t}{\epsilon}\right)-U^0(x,t)\right|=0. \tag{5.47}$$

Then, (1.18) follows from (5.45)–(5.47) immediately.

From (1.5), it is clear that

$$U^0(x,t)=U^0\left(\frac{x}{\epsilon},\frac{t}{\epsilon}\right). \tag{5.48}$$

By (5.45) and (5.48), (1.20) follows directly from (5.39), Lemma 3.1 and Sobolev inequality.

Therefore, the proof of Theorem 1.1 is completed. □

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