

# COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH TEMPERATURE DEPENDENT HEAT CONDUCTIVITIES

RONGHUA PAN AND WEIZHE ZHANG

ABSTRACT. We prove the existence and uniqueness of global strong solutions to the one dimensional, compressible Navier-Stokes system for the viscous and heat conducting ideal polytropic gas flow, when heat conductivity depends on temperature in power law of Chapman-Enskog. The results reported in this article is valid for initial boundary value problem with non-slip and heat insulated boundary along with smooth initial data with positive temperature and density without smallness assumption.

**Keywords:** Compressible Navier-Stokes equations, Global Strong Solutions, Uniqueness, Temperature dependent heat conductivity

## 1. INTRODUCTION

Consider the following initial boundary value problem for compressible Navier-Stokes equations in one space dimension and in the Lagrangian coordinates:

$$\left\{ \begin{array}{l} v_t - u_x = 0, \quad x \in (0, 1), t > 0, \\ u_t + p_x = \left(\frac{\mu u_x}{v}\right)_x, \\ \left(e + \frac{1}{2}u^2\right)_t + (pu)_x = \left[\frac{(\kappa\theta_x + \mu u u_x)}{v}\right]_x, \\ (v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x), x \in [0, 1] \\ u(0, t) = u(1, t) = 0, \quad \theta_x(0, t) = \theta_x(1, t) = 0, \quad t \geq 0. \end{array} \right. \quad (1)$$

Here  $x$  is the Lagrangian space variable, and  $t$  is the time.  $v$ ,  $u$ ,  $\theta$  are specific volume, fluid velocity and absolute temperature respectively. In general, the pressure  $p$ , specific internal energy  $e$ , viscosity  $\mu$  and heat conductivity  $\kappa$  are functions of  $\theta$  and  $v$ . The Gibb's equation

$$\theta \, dS = de + p \, dv$$

gives the relation of  $e, p, v, S$ , where  $S$  is the specific entropy. In this paper, we focus on ideal polytropic gas and the constitution relation reads

$$p(\theta, v) = \frac{K\theta}{v}, \quad e = c_v\theta, \quad (2)$$

where  $K$  and  $c_v$  are positive constants.

While in classical literatures, many results have been obtained for constant viscosity and heat conductivity coefficients, we will focus on the case when they depend on temperature in a physical way. In fact, when one considers the first level (Chapman-Enskog) approximation in

kinetic theory, viscosity and heat conductivity are functions of temperature. According to [2], the following relations hold

$$\mu = \bar{\mu}\theta^b, \quad \kappa = \bar{\kappa}\theta^b, \quad b \in \left(\frac{1}{2}, \infty\right). \quad (3)$$

Here  $\bar{\mu}$  and  $\bar{\kappa}$  are two positive constants. Indeed, if the intermolecular potential varies as  $r^{-a}$ , for  $r$  the intermolecular distance, then the constant  $b = \frac{a+4}{2a}$ , see [2]. In particular, for Maxwellian molecules  $a = 4$ , and  $b = 1$ ; while for elastic spheres,  $a = \infty$ , and  $b = \frac{1}{2}$ .

In the classical setting when  $\mu$  and  $\kappa$  are constants, the existence of strong solutions to (1) has been successfully studied by many mathematicians, both local and global theories were established long ago, see [21],[14] and [24], for local theory, and [17] for global theory. Such results have been further generalized to nonlinear thermoviscoelasticity by [4], and [5], and to viscous heat-conductive “real gases” by [16], [20], [12], and [22]. In either case,  $\mu$  is independent of  $\theta$ , and heat conductivity is allowed to depend on temperature in a special way with a positive lower bound, and balanced with corresponding constitution relations. We refer the readers to [1], [9], and [10] for more references and some recent discussions.

In contrast to the fruitful development of the classical setting, the treatment on the physical case (3) is lacking due to the new difficulty introduced in such relations, which lead to possible degeneracy and strong nonlinearity in viscosity and heat diffusion. As a first step in this direction, [13] proved the global existence of a weak solution to (1) under the assumption

$$\mu = \bar{\mu}, \quad \kappa = \bar{\kappa}\theta^b, \quad \text{for } b \in \left[0, \frac{3}{2}\right).$$

Although this simplified relation has constant viscosity, the porous medium type diffusion in energy equation introduced some significant difficulties. [13] made extra efforts to overcome these difficulties to achieve the global existence of weak solutions. Unfortunately, the estimates obtained in [13] are not sufficient to prove the existence of global strong solutions or classical solutions. Our mission in this paper is to establish the global existence of strong and/or classical solutions to (1) under a milder assumption on viscosity and heat conductivity than [13], that is,

$$\mu = \bar{\mu}, \quad \kappa = \bar{\kappa}\theta^b \quad b \in [0, \infty). \quad (4)$$

Under this assumption (4), we will show that (1) admits a unique global strong solution for the problem when the initial data  $(v_0, u_0, \theta_0)(x)$  is in  $H^1 \times H^2 \times H^2$ . The solution will be classical with a better initial data. More precisely, our main result is

**Theorem 1.1.** *Suppose that  $\mu$  and  $\kappa$  satisfy (4) for some positive constants  $\bar{\mu}$  and  $\bar{\kappa}$ . If the initial data  $(v_0, u_0, \theta_0)(x)$  is compatible with the boundary conditions, satisfying*

$$(v_0, u_0, \theta_0)(x) \in H^1 \times H^2 \times H^2, \quad \int_0^1 v_0(x) dx = M > 0, \quad (5)$$

and there are constants  $\underline{v}, \bar{v}, \underline{\theta}, \bar{\theta}$  such that

$$0 < \underline{v} \leq v_0(x) \leq \bar{v}, \quad 0 < \underline{\theta} \leq \theta_0(x) \leq \bar{\theta}, \quad (6)$$

then (1) admits a unique global strong solution  $(v, u, \theta)(x, t)$  such that, for any fixed  $T > 0$ ,

$$v \in L^\infty([0, T]; H^1([0, 1])), \quad u \in L^\infty([0, T]; H^2([0, 1])), \quad \theta \in L^\infty([0, T]; H^2([0, 1])), \quad (7)$$

and for each  $(x, t) \in [0, 1] \times [0, T]$ ,

$$\begin{cases} C^{-1} \leq v(x, t) \leq C, \quad C^{-1} \leq \theta(x, t) \leq C, \quad |u(x, t)| \leq C, \\ \|(v, u, \theta)(\cdot, t)\|_{H^1}^2 + \int_0^t \|(v, u, \theta)(\cdot, s)\|_{H^1}^2 ds \leq C, \\ \|(u, \theta)(\cdot, t)\|_{H^2}^2 + \int_0^t \|(v_{xt}, u_{xx}, u_{xt}, \theta_{xx}, \theta_{xt})(\cdot, s)\|_{L^2}^2 ds \leq C, \end{cases} \quad (8)$$

where  $C > 0$  is some constant depending on initial data and  $T$ .

If in addition the initial data  $(v_0, u_0, \theta_0)(x)$  satisfies

$$v_0(x) \in C^{1+\alpha}, \quad u_0(x) \in C^{2+\alpha}, \quad \theta_0(x) \in C^{2+\alpha}, \quad (9)$$

for some  $\alpha \in (0, 1)$ , then (1) has a unique global classical solution  $(v, u, \theta)(x, t)$  such that, for any fixed  $T > 0$ ,

$$v \in C^{1+\alpha, \frac{\alpha}{2}}([0, 1] \times [0, T]), \quad u \in C^{2+\alpha, 1+\frac{\alpha}{2}}([0, 1] \times [0, T]), \quad \theta \in C^{2+\alpha, 1+\frac{\alpha}{2}}([0, 1] \times [0, T]), \quad (10)$$

and for each  $(x, t) \in [0, 1] \times [0, T]$ ,  $(v, u, \theta)(x, t)$  satisfies (8).

The existence and uniqueness of local in time solution can be obtained by a standard Banach fixed point argument due to the contraction of the solution operators defined by the linearized problem, c.f. [21], [14] and [24]. As a special case of the result in [24], the following lemma gives the local existence for the purpose of our problem.

**Lemma 1.1.** *If (4) and (5), (6) and (7) hold, and the initial data is compatible with boundary conditions, then there exists a unique local strong solution to (1)  $(v, u, \theta)(x, t)$  on  $[0, 1] \times [0, T_1]$  for some  $T_1 > 0$  depending on the initial data, such that (8) holds for  $t \in [0, T_1]$ . If the initial data further satisfies (9), then  $v \in C^{1+\alpha, \frac{\alpha}{2}}([0, 1] \times [0, T_1])$ ,  $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}([0, 1] \times [0, T_1])$ , and  $\theta \in C^{2+\alpha, 1+\frac{\alpha}{2}}([0, 1] \times [0, T_1])$ .*

Based on the local existence result, the existence of global solution will be established by extending the local solution in time with the help of the global a priori estimates stated in (8). It is clear that (8) is sufficient to extend the local strong solution to global one. For the classical solution, with the help of the better initial data (9), estimates in (8) is indeed sufficient to derive the Hölder estimates (10), following the standard method and argument, see [14], [17], or [18]. Therefore, in order to prove Theorem 1.1, it remains to obtain the key a priori estimates (8), which will be carried out in the Section 2.

We will outline the main ideas and difficulties in our problem comparing to previous results. In principle, we will follow the basic framework laid out in [4] with extra attention to the new difficulties from the porous medium type diffusion in the energy equation. Unlike the ‘‘real gas’’ case, where  $\kappa$  has positive lower bound along with some fine structure of the growth which is well-balanced with corresponding constitution relations, relation (4) with pure ideal gas constitution

relations in our case introduced some significant difficulties in several ways. We will discuss two main issues.

First, unless  $b = 0$ , relation (4) permits a possible degeneracy in heat diffusion. In fact, the viscosity or heat diffusion itself is not sufficient to prevent the break down of solutions from smooth initial data, see [6]. However, we observe that in this setting (4),  $\kappa$  vanishes only when  $\theta = 0$ . According to the third law of thermodynamics, the absolute temperature  $\theta$  is expected to be positive for all the time. Therefore, the positivity of  $\theta$  is not only an important issue in proving our theorem, but also the physical justification of the model. In Lemma 2.2 below, we will show that if the initial absolute temperature is positive, then it stays positive for any finite time. This laid a firm foundation for our further development.

The other key issue lies on the extra nonlinearity due to (4) when  $b > 0$ . We emphasize here that unlike the case of “real gas”, the constitution relations in this paper do not balance with the growth of  $\kappa$  in  $\theta$ . Therefore, much attentions were paid in order to control this strong nonlinearity. In particular, the usual entropy dissipation estimates is not enough, we proved the new refined estimates on  $\theta$  in Lemma 2.2, and Lemma 2.4, which plays some key roles in the later development. Even with the help of these new estimates, new trouble appears in the higher order estimates. It turns out that one has to separate the cases of  $b < 1$  from  $b \geq 1$ . The behaviors of solutions in these two cases are quite different, see Lemma 2.8 and Lemma 2.12 below. Furthermore, the case when  $b = 1$  requires some extra efforts. A critical new estimates is obtained with the help of a new functional introduced in (85).

We remark that, although we successfully solved (1) under the condition (4), it is still far away from a theory for (1) under the condition (3), which remains as an interesting open problem. For a local existence theory in some Besov space, and global existence with initial data small in some Besov norm, we refer to [11] for the Cauchy problem in three space dimensions.

## 2. A PRIORI ESTIMATES

We will derive the estimates for the solutions in this section to prove Theorem 1.1. Throughout the rest of this paper, we will assume  $b > 0$ , since the case when  $b = 0$  was solved in [17]. Due to the complicated structure and nonlinearity in the system along with the difficulty from the large initial data, the estimate is somewhat delicate. We will perform a sequence of estimates which are stated in the following as lemmas. For simplicity of presentation, we will fix  $\bar{\mu} = \bar{\kappa} = K = c_v = 1$ . We now assume that  $(v, u, \theta)(x, t)$  is the unique strong solution of (1) defined on  $[0, 1] \times [0, T]$  for some  $T > 0$ . In the following, unless specified,  $C = C(T)$  denotes the generic constant which may different from line to line.

First of all, we have the following conservation laws of mass and energy from (1),

$$\int_0^1 v(x, t) dx = \int_0^1 v_0(x) dx = M, \quad (11)$$

$$\int_0^1 \left( \theta + \frac{1}{2} u^2 \right) (x, t) dx = C. \quad (12)$$

Thanks to the constant viscosity coefficient, we follow the approach of [17] to obtain the following lower bound of the specific volume,

**Lemma 2.1.** *Under the conditions (4)–(6), There exists a positive constant  $C > 0$  such that, for any  $(x, t) \in [0, 1] \times [0, T]$*

$$C^{-1} \leq v(x, t) \leq C + C \int_0^T \max_{x \in [0, 1]} \theta(x, s) ds. \quad (13)$$

*Proof.* From (11) and mean value theorem, for each time  $t \in [0, T]$ , there exists a  $x_0(t) \in [0, 1]$ , such that  $v(x_0(t), t) = M$ . Using the mass equation, one can rewrite the momentum equation as

$$u_t = (\ln v)_{xt} - \left(\frac{\theta}{v}\right)_x.$$

Integrating this equation in time over  $[0, t]$  for any  $t \in [0, T]$ , and then integrating in space from  $x_0(t)$  to  $x$ , we get

$$\begin{aligned} \int_{x_0(t)}^x u(\xi, t) - u(\xi, 0) d\xi &= \ln v(x, t) - \ln v(x, 0) - \ln v(x_0(t), t) + \ln v(x_0(t), 0) \\ &\quad - \int_0^t \left[ \frac{\theta}{v}(x, s) - \frac{\theta}{v}(x_0(t), s) \right] ds \end{aligned} \quad (14)$$

Defining

$$B_1(t) = e^{\int_0^t \frac{\theta}{v}(x_0(t), s) ds}, \quad (15)$$

$$B_2(x, t) = e^{\int_{x_0(t)}^x u(\xi, 0) - u(\xi, t) d\xi}, \quad (16)$$

taking exponential on both side of (14), we then get

$$\frac{e^{\int_0^t \frac{\theta}{v}(x, s) ds}}{v(x, t)} = \frac{v(x_0(t), 0)}{Mv(x, 0)} B_1(t) B_2(x, t) \quad (17)$$

Multiplying (17) with  $\theta$  and integrating in time, we get the relation

$$e^{\int_0^t \frac{\theta}{v}(x, s) ds} = 1 + \int_0^t \frac{v(x_0(s), 0)}{Mv(x, 0)} B_1(s) B_2(x, s) \theta(x, s) ds \quad (18)$$

Plugging (18) into (17), one obtains

$$v(x, t) = \frac{Mv(x, 0) \left( 1 + \int_0^t \frac{v(x_0(s), 0)}{Mv(x, 0)} B_1(s) B_2(x, s) \theta(x, s) ds \right)}{v(x_0(t), 0) B_1(t) B_2(x, t)} \quad (19)$$

First, there exists  $C$  such that

$$\left| \int_{x_0(t)}^x u(\xi, 0) - u(\xi, t) d\xi \right| \leq \left( 2 \int_0^1 u^2(x, t) dx \right)^{\frac{1}{2}} \leq C, \quad (20)$$

$$\int_0^t \frac{\theta}{v}(x_0(t), s) ds \geq 0. \quad (21)$$

Then there exists a constant  $C > 0$  such that

$$C^{-1} \leq B_2 \leq C \text{ and } C^{-1} \leq B_1. \quad (22)$$

For the upper bound of  $B_1$ . We observe from (19) that

$$v(x, t)B_1(t) = \frac{Mv(x, 0)(1 + \int_0^t \frac{v(x_0(s), 0)}{Mv(x, 0)} B_1(s)B_2(x, s)\theta(x, s) ds)}{v(x_0(t), 0)B_2(x, t)} \quad (23)$$

Now integrating (23) in  $x$  from 0 to 1, we find

$$\begin{aligned} B_1(t) &= \int_0^1 \frac{v(\xi, 0)}{v(x_0(t), 0)B_2(\xi, t)} d\xi \\ &+ \int_0^1 \int_0^t \frac{v(x_0(s), 0)}{Mv(x_0(t), 0)B_2(\xi, t)} B_1(s)B_2(\xi, s)\theta(\xi, s) ds d\xi \\ &\leq C + C \int_0^t B_1(s) ds \end{aligned} \quad (24)$$

By the Gronwall inequality, there exists a constant  $C$  such that

$$B_1(t) \leq C. \quad (25)$$

Now, we read from (19) that

$$v(x, t) \geq \frac{Mv(x, 0)}{v(x_0(t), 0)B_1(t)B_2(x, t)} \geq C^{-1},$$

for some positive constant  $C$ . Now, (19) implies that

$$v(x, t) \leq C + C \int_0^t \theta(x, s) ds.$$

This completes the proof of this lemma.  $\square$

With the help of constitution relations (2) and (4), we can deduce the following temperature equation from the system (1),

$$\theta_t = \left(\frac{\theta^b \theta_x}{v}\right)_x + \frac{u_x^2}{v} - \frac{\theta}{v} u_x. \quad (26)$$

According to the third law of thermodynamics, it is expected that the absolute temperature  $\theta$  stays positive all the time. We will show this result in the following lemma. This result is very important for us since the heat conductivity degenerates at  $\theta = 0$ . We now prove

**Lemma 2.2.** *There exists a constant  $C$  such that, for any  $p > 2$ ,*

$$\left\| \frac{1}{\theta} \right\|_{L^{p-1}} \leq C, \quad \int_0^T \int_0^1 \left( \frac{u_x^2}{v\theta^p} + \frac{\theta^b \theta_x^2}{v\theta^{p+1}} \right) dx dt \leq C, \quad (27)$$

and in particular,

$$0 < C^{-1} \leq \theta(x, t), \quad \forall (x, t) \in [0, 1] \times [0, T]. \quad (28)$$

*Proof.* For any  $p > 2$ , we multiply (26) by  $1/\theta^p$ , integrate over  $[0, 1]$ , integrate by parts, and use (13), we get

$$\begin{aligned} & \frac{1}{p-1} \frac{d}{dt} \int_0^1 \frac{1}{\theta^{p-1}} dx + \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^{p+1}} dx + \int_0^1 \frac{u_x^2}{v \theta^p} dx = \int_0^1 \frac{\theta^{1-p} u_x}{v} dx \\ & \leq \frac{1}{2} \int_0^1 \frac{u_x^2}{v \theta^p} dx + 2 \int_0^1 \frac{1}{v \theta^{p-2}} dx \\ & \leq \frac{1}{2} \int_0^1 \frac{u_x^2}{v \theta^p} dx + C \left( \int_0^1 \frac{1}{\theta^{p-1}} dx \right)^{\frac{p-2}{p-1}}, \end{aligned} \quad (29)$$

where we have used the lower bound of  $v$  and Hölder inequality. (29) implies that

$$\left( \frac{d}{dt} \left\| \frac{1}{\theta} \right\|_{L^{p-1}} \right) \left( \int_0^1 \frac{1}{\theta^{p-1}} dx \right)^{\frac{p-2}{p-1}} \leq C \left( \int_0^1 \frac{1}{\theta^{p-1}} dx \right)^{\frac{p-2}{p-1}}, \quad (30)$$

which is

$$\frac{d}{dt} \left\| \frac{1}{\theta} \right\|_{L^{p-1}} \leq C, \text{ or } \left\| \frac{1}{\theta} \right\|_{L^{p-1}} \leq C. \quad (31)$$

Substituting this estimate back to (29), integrating in time, it gives

$$\int_0^T \int_0^1 \left( \frac{u_x^2}{v \theta^p} + \frac{\theta^b \theta_x^2}{v \theta^{p+1}} \right) dx dt \leq C.$$

The constant  $C$  in (31) depends only in time and initial data. Leting  $p$  go to  $\infty$ , we proved Lemma 2.2.  $\square$

In order to explore the dissipation mechanism in compressible Navier-Stokes equations, one usually turns to second law of themodynamics. To this end, we recall Gibb's equation

$$\theta dS = de + pdv.$$

With the help of constitution relation we imposed in (2), one can choose the specific entropy

$$S = \ln \theta + \ln v. \quad (32)$$

With help of equation (1) and (26), it is clear that

$$\begin{aligned} S_t &= \frac{\theta_t}{\theta} + \frac{v_t}{v} \\ &= \frac{1}{\theta} \left( \frac{\theta^b \theta_x}{v} \right)_x + \frac{u_x^2}{v \theta} - \frac{1}{v} u_x + \frac{1}{v} v_t \\ &= \left( \frac{\theta^b \theta_x}{v \theta} \right)_x + \frac{\theta^b \theta_x^2}{v \theta^2} + \frac{u_x^2}{v \theta} \end{aligned} \quad (33)$$

We now prove the following entropy estimate by (33).

**Lemma 2.3.** *There exists a constant  $C > 0$  such that*

$$\int_0^T \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^2} dx dt + \int_0^T \int_0^1 \frac{u_x^2}{v \theta} dx dt + \sup_{0 \leq t \leq T} \int_0^1 |\ln \theta| + |\ln v| dx \leq C. \quad (34)$$

and there exists  $a(t) \in \Omega$  and a constant  $C$ , such that

$$\theta(a(t), t) \leq C. \quad (35)$$

*Proof.* With the help of (11) and (12), integrating (33) over  $[0, 1] \times [0, t]$ , we get

$$\int_0^T \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^2} dx dt + \int_0^T \int_0^1 \frac{u_x^2}{v \theta} dx dt - \int_0^1 s(x, t) dx \leq C - \int_0^1 s(x, 0) dx \quad (36)$$

where

$$s = \ln \theta - \theta + 1 + \ln v - v + 1. \quad (37)$$

We note that  $-s(\theta, v) \geq 0$  is a convex function on  $\theta$  and  $v$ , and that

$$0 \leq - \int_0^1 s(x, 0) dx \leq C,$$

for some positive constant  $C$ . Therefore, it holds that

$$\int_0^T \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^2} dx dt + \int_0^T \int_0^1 \frac{u_x^2}{v \theta} dx dt \leq C,$$

and

$$\int_0^1 -s(t) dx \leq C.$$

Which, with the help of (11) and (12), implies that

$$\sup_{0 \leq t \leq T} \int_0^1 |\ln \theta| + |\ln v| dx \leq C.$$

On the other hand, from (12) we have

$$\int_0^1 \theta(x, t) dx \leq C \quad (38)$$

By the mean value theorem, there exists  $a(t) \in [0, 1]$ , such that

$$\theta(a(t), t) = \int_0^1 \theta(x, t) dx \leq C. \quad (39)$$

□

When both  $\mu$  and  $\kappa$  are constants (or have leading constant terms), the dissipation estimates provided in the entropy estimate are usually enough for future development. However, in our case, the nonlinearity in  $\kappa$  on  $\theta$  requires further attention on the control of  $\theta$ . For this purpose, one of our main ingredients in this paper is the following refined estimates on temperature.

**Lemma 2.4.** *For any constant  $\epsilon \in (0, 1)$ , there exists a constant  $C > 0$  such that*

$$\int_0^T \int_0^1 \theta^{3+b-\epsilon} dx dt + \int_0^T \max_{x \in [0, 1]} \theta^{2+b-\epsilon} dt + \int_0^T \int_0^1 \frac{\theta_x^2 \theta^b}{v \theta^{1+\epsilon}} dx dt \leq C. \quad (40)$$



*Proof.* For any  $p \in (0, 1)$ , multiplying (26) with  $1/\theta^p$ , integrating by parts, it turns out

$$\frac{1}{1-p} \frac{d}{dt} \int_0^1 \theta^{1-p} dx = p \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^{p+1}} dx + \int_0^1 \frac{u_x^2}{v \theta^p} dx - \int_0^1 \frac{\theta^{1-p} u_x}{v} dx \quad (41)$$

Using (12), Lemma 2.2, and the fact  $1-p \in (0, 1)$ , it is clear that

$$\int_0^1 \theta^{1-p}(x, t) dx \leq C \int_0^1 \theta dx \leq C.$$

We now integrate (41) in time, and use Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned} & \int_0^T \int_0^1 \frac{p \theta^b \theta_x^2}{v \theta^{p+1}} dx dt + \int_0^T \int_0^1 \frac{u_x^2}{v \theta^p} dx dt \\ & \leq C + \int_0^T \int_0^1 \frac{\theta^{1-p} u_x}{v} dx dt \\ & \leq \frac{1}{2} \int_0^T \int_0^1 \frac{u_x^2}{v \theta^p} dx dt + 2 \int_0^T \int_0^1 \frac{\theta^{2-p}}{v} dx dt + C, \end{aligned} \quad (42)$$

which implies

$$\int_0^T \int_0^1 \frac{p \theta^b \theta_x^2}{v \theta^{p+1}} dx dt + \int_0^T \int_0^1 \frac{u_x^2}{v \theta^p} dx dt \leq C + C \int_0^T \max_{x \in [0,1]} \theta^{1-p} dt. \quad (43)$$

With the help of Cauchy-Schwartz inequality, and Lemma 2.2, we further have the following estimate

$$\begin{aligned} \int_0^T \max_{x \in [0,1]} \theta^{1-p} dt & \leq C + \int_0^T \max_{x \in [0,1]} (|\theta^{1-p} - (\theta(a(t), t))^{1-p}|) dt \\ & \leq C + (1-p) \int_0^T \int_0^1 |\theta^{-p} \theta_x| dx dt \\ & \leq C + \delta \int_0^T \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^{p+1}} dx dt + C(\delta) \int_0^T \int_0^1 v \theta^{1-b-p} dx dt \\ & \leq C + \frac{p}{2} \int_0^T \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^{p+1}} dx dt + C \int_0^T (\max_{x \in [0,1]} \theta^{1-b-p}) \int_0^1 v dx dt \\ & \leq \delta \int_0^T \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^{p+1}} dx dt + \frac{1}{2} \int_0^T \max_{x \in [0,1]} \theta^{1-p} dt + C(\delta), \end{aligned} \quad (44)$$

for  $\delta > 0$  small enough. Now, we conclude from (43) and (44) that, for any constant  $0 < p < 1$ , there exists a constant  $C > 0$ , such that

$$\int_0^T \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^{p+1}} dx dt + \int_0^T \int_0^1 \frac{u_x^2}{v \theta^p} dx dt \leq C. \quad (45)$$

Recall  $a(t)$  defined in Lemma 2.3, we finally have for any  $\epsilon \in (0, 1)$  that

$$\begin{aligned}
\int_0^T \int_0^1 \theta^{3+b-\epsilon} dx dt &\leq \int_0^T \max_{x \in [0,1]} \theta^{2+b-\epsilon} dt \\
&\leq C + C \int_0^T \left( \max_{x \in [0,1]} |\theta^{\frac{2+b-\epsilon}{2}}(x, t) - \theta^{\frac{2+b-\epsilon}{2}}(a(t), t)| \right)^2 dt \\
&\leq C + C \int_0^T \left( \int_0^1 |\theta^{\frac{b-\epsilon}{2}} \theta_x| dx \right)^2 dt \\
&\leq C + C \int_0^T \left( \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^{\epsilon+1}} dx \right) \left( \int_0^1 v \theta dx \right) dt \\
&\leq C + C \left( \max_{(x,t) \in [0,1] \times [0,T]} v(x, t) \right) \int_0^T \int_0^1 \frac{\theta^b \theta_x^2}{v \theta^{\epsilon+1}} dx dt \\
&\leq C + C \int_0^T \max_{x \in [0,1]} \theta(x, t) dt,
\end{aligned} \tag{46}$$

which implies that

$$\int_0^T \max_{x \in [0,1]} \theta^{2+b-\epsilon} dt \leq C + \frac{1}{2} \int_0^T \max_{x \in [0,1]} \theta(x, t)^{2+b-\epsilon} dt,$$

and therefore

$$\int_0^T \max_{x \in [0,1]} \theta^{2+b-\epsilon} dt \leq C. \tag{47}$$

(46) and (47) complete the proof of this lemma.  $\square$

By Lemma 2.1, estimates in (46) and (47) also give the upper bound of  $v$ .

**Lemma 2.5.** *There is a positive constant  $C$ , such that*

$$\max_{(x,t) \in [0,1] \times [0,T]} v(x, t) \leq C.$$

As another direct consequence, we have the following estimates on velocity,

**Lemma 2.6.** *There exists  $C$  such that*

$$\sup_{0 \leq t \leq T} \int_0^1 u^2 dx + \int_0^T \int_0^1 u_x^2 dx dt \leq C \tag{48}$$

*Proof.* We multiply the momentum equation with  $u$ , then integrate the result over  $[0, 1] \times [0, T]$ , and integrate by parts to obtain

$$\begin{aligned}
\sup_{0 \leq t \leq T} \int_0^1 u^2 dx + \int_0^T \int_0^1 u_x^2 dx dt &\leq C + C \int_0^T \int_0^1 p^2 dx dt \\
&\leq C + C \int_0^T \int_0^1 \theta^2 dx dt \\
&\leq C.
\end{aligned} \tag{49}$$

$\square$

The next lemma gives estimates on the  $L^2$  norm of  $v_x$ .

**Lemma 2.7.** *There exists a constant  $C > 0$  such that*

$$\sup_{0 \leq t \leq T} \int_0^1 v_x^2(x, t) dx \leq C. \quad (50)$$

*Proof.* Since

$$\left(\frac{v_t}{v}\right)_x = \left(\frac{v_x}{v}\right)_t = (\ln v)_{xt} \quad (51)$$

one can rewrite momentum equation as

$$\left(\frac{v_x}{v}\right)_t = u_t + \left(\frac{\theta}{v}\right)_x \quad (52)$$

Multiplying this equation (52) by  $\frac{v_x}{v}$ , we reach

$$\begin{aligned} \frac{1}{2} \left[ \left(\frac{v_x}{v}\right)^2 \right]_t &= \frac{v_x}{v} u_t + \frac{v_x}{v} \left(\frac{\theta}{v}\right)_x \\ &= \left(\frac{v_x}{v} u\right)_t - (\ln v)_{xt} u + \frac{v_x \theta_x}{v^2} - \frac{v_x^2 \theta}{v^3}, \end{aligned} \quad (53)$$

where

$$-(\ln v)_{xt} u = -[(\ln v)_t u]_x + \frac{u_x^2}{v}.$$

We now integrate (53) over  $[0, 1] \times [0, T]$ . After re-collecting terms, one has

$$\begin{aligned} &\sup_{0 \leq t \leq T} \int_0^1 \left[ \frac{1}{2} \left(\frac{v_x}{v}\right)^2 + \frac{v_x}{v} u \right] dx + \int_0^T \int_0^1 \frac{v_x^2 \theta}{v^3} dx dt \\ &\leq C + \int_0^T \int_0^1 \frac{v_x \theta_x}{v^2} dx dt \\ &\leq C + C \int_0^T \int_0^1 \frac{v_x^2 \theta^{2-b}}{v^2} dx dt + C \int_0^T \int_0^1 \frac{\theta_x^2 \theta^b}{\theta^2} dx dt \\ &\leq C + C \int_0^T \int_0^1 \frac{v_x^2 \theta^2}{v^2} dx dt \\ &\leq C + C \int_0^T (\max_{x \in [0,1]} \theta^2) \left( \int_0^1 \frac{v_x^2}{v^2} dx \right) dt. \end{aligned} \quad (54)$$

Noting that

$$\int_0^1 \frac{v_x}{v} u dx \leq \frac{1}{4} \int_0^1 \left(\frac{v_x}{v}\right)^2 dx + 4 \int_0^1 u^2 dx,$$

we infer from (54)

$$\sup_{0 \leq t \leq T} \int_0^1 \left(\frac{v_x}{v}\right)^2 dx \leq C + C \int_0^t (\max_{x \in [0,1]} \theta^2) \left( \int_0^1 \frac{v_x^2}{v^2} dx \right) dt, \quad (55)$$

which, along with Gronwall inequality and Lemma 2.4, gives the estimates in lemma 2.7  $\square$

In order to obtain the higher order estimates, we will follow the framework introduced in [4], and define the following two functionals

$$Z(t) = \sup_{0 \leq t \leq T} \int_0^1 u_{xx}^2(x, t) dx, \quad Y(t) = \sup_{0 \leq t \leq T} \int_0^1 \theta^{2b} \theta_x^2 dx. \quad (56)$$

These two functionals will be useful to select out simple rules behind the tangled relations of the higher order norms and the upper bound of  $\theta$ . First of all, the following relations hold.

**Lemma 2.8.** *There exists a constant  $C > 0$  such that*

$$\max_{(x,t) \in [0,1] \times [0,T]} |u_x| \leq C + CZ^{\frac{3}{8}}, \quad (57)$$

$$\max_{(x,t) \in [0,1] \times [0,T]} \theta \leq C + CY^{\frac{1}{3+b}}. \quad (58)$$

*Proof.* The first estimate is the same as that in [4]. In fact, noting

$$\int_0^1 u_x dx = 0,$$

one has

$$\max_{x \in [0,1]} u_x^2 \leq 2 \left( \int_0^1 u_x^2 dx \right)^{\frac{1}{2}} \left( \int_0^1 u_{xx}^2 dx \right)^{\frac{1}{2}}.$$

This estimate, together with the following standard interpolation estimate

$$\int_0^1 u_x^2 dx \leq C \int_0^1 u^2 dx + C \left( \int_0^1 u^2 dx \right)^{\frac{1}{2}} \left( \int_0^1 u_{xx}^2 dx \right)^{\frac{1}{2}},$$

gives

$$\max_{x \in [0,1]} u_x^2 \leq C + CZ^{\frac{3}{4}},$$

which implies the first assertion in the lemma.

We now focus on the second one. Based on (12), recall  $a(t)$  defined in Lemma 2.3, we have

$$\begin{aligned} \max_{(x,t) \in [0,1] \times [0,T]} \theta^{3+b} &= C + \max_{(x,t) \in [0,1] \times [0,T]} \left( \theta^{\frac{3+b}{2}} - (\theta(a(t), t))^{\frac{3+b}{2}} \right)^2 \\ &\leq C + \sup_{0 \leq t \leq T} \left( \int_0^1 \theta^{\frac{1+b}{2}} \theta_x dx \right)^2 \\ &\leq C + \sup_{0 \leq t \leq T} \left( \int_0^1 \theta^{2b} \theta_x^2 dx \right) \left( \int_0^1 \theta^{1-b} dx \right) \\ &\leq C + CY, \end{aligned} \quad (59)$$

which implies (58).  $\square$

The power type nonlinearity in heat conductivity on  $\theta$  exhibits different behaviors when  $b$  changes. For this purpose, we will use different strategies for the case when  $b < 1$  and for the case when  $b \geq 1$ . We first deal with the case when  $b < 1$ .

**Lemma 2.9.** *When  $0 < b < 1$  there exists a constant  $C$  and a constant  $C(b) \in (0, 1)$  depending only on  $b$  such that*

$$\int_0^T \int_0^1 \theta^b \theta_t^2 dx dt + \sup_{0 \leq t \leq T} \int_0^1 \theta^{2b} \theta_x^2 dx \leq C + CZ^{C(b)}. \quad (60)$$

*Proof.* Define

$$K(\theta, v) = \frac{\theta^{1+b}}{(1+b)v}. \quad (61)$$

Then we compute

$$K_t = \frac{\theta^b \theta_t}{v} - \frac{\theta^{1+b} v_t}{(1+b)v^2} \quad (62)$$

$$K_x = \frac{\theta^b \theta_x}{v} - \frac{\theta^{1+b} v_x}{(1+b)v^2} \quad (63)$$

$$\begin{aligned} K_{xt} &= \left(\frac{\theta^b \theta_x}{v}\right)_t + \frac{2\theta^{1+b} v_x u_x}{(1+b)v^3} - \frac{\theta^b \theta_t v_x}{v^2} - \frac{\theta^{1+b} u_{xx}}{(1+b)v^2} \\ &=: \left(\frac{\theta^b \theta_x}{v}\right)_t + \tilde{K}. \end{aligned} \quad (64)$$

Multiplying  $K_t$  on both sides of the temperature equation (26) and integrating by parts over  $[0, 1]$  in  $x$ , using the above identities (62), (63), and (64), we get

$$\begin{aligned} &\int_0^1 \frac{\theta^b \theta_t^2}{v} dx - \int_0^1 \frac{\theta^{1+b} \theta_t v_t}{(1+b)v^2} dx \\ &= - \int_0^1 \left(\frac{\theta^b \theta_x}{v}\right) K_{xt} dx + \int_0^1 \left(\frac{u_x^2}{v} - \frac{\theta}{v} u_x\right) K_t dx \\ &= -\frac{1}{2} \frac{d}{dt} \int_0^1 \left(\frac{\theta^b \theta_x}{v}\right)^2 dx - \int_0^1 \frac{\theta^b \theta_x}{v} \tilde{K} dx + \int_0^1 \left(\frac{u_x^2}{v} - \frac{\theta}{v} u_x\right) K_t dx, \end{aligned} \quad (65)$$

which implies that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_0^1 \left(\frac{\theta^b \theta_x}{v}\right)^2 dx + \int_0^1 \frac{\theta^b \theta_t^2}{v} dx \\ &= \int_0^1 \frac{\theta^{1+b} \theta_t u_x}{(1+b)v^2} dx - \int_0^1 \frac{\theta^b \theta_x}{v} \tilde{K} dx + \int_0^1 \left(\frac{u_x^2}{v} - \frac{\theta}{v} u_x\right) K_t dx. \end{aligned} \quad (66)$$

We now integrate (66) from 0 to  $t$ , it turns out to

$$\begin{aligned} &\frac{1}{2} \int_0^1 \left(\frac{\theta^b \theta_x}{v}\right)^2 dx + \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt \\ &= C + \int_0^t \int_0^1 \frac{\theta^{1+b} \theta_t u_x}{(1+b)v^2} dx dt - \int_0^t \int_0^1 \frac{\theta^b \theta_x}{v} \tilde{K} dx dt + \int_0^t \int_0^1 \left(\frac{u_x^2}{v} - \frac{\theta}{v} u_x\right) K_t dx dt. \end{aligned} \quad (67)$$

We will now estimate the last three terms in (67). The first term can be estimated as follows,

$$\begin{aligned} &\left| \int_0^t \int_0^1 \frac{\theta^{1+b} \theta_t u_x}{(1+b)v^2} dx dt \right| \\ &\leq \frac{1}{2} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \int_0^t \int_0^1 \frac{\theta^{2+b} u_x^2}{v^3} dx dt \\ &\leq \frac{1}{2} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \max_{(x,t) \in [0,1] \times [0,T]} |u_x|^2 \int_0^t \int_0^1 \theta^{2+b} dx dt \\ &\leq \frac{1}{2} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + CZ^{\frac{3}{4}} + C, \end{aligned} \quad (68)$$

where we have used the bounds of  $v$ , and Lemmas 2.4 and 2.8. We now turn to

$$\begin{aligned} \left| \int_0^t \int_0^1 \frac{\theta^b \theta_x}{v} \tilde{K} \, dx dt \right| &= \left| \int_0^t \int_0^1 \frac{\theta^b \theta_x}{v} \left[ \frac{2\theta^{1+b} v_x u_x}{(1+b)v^3} - \frac{\theta^b \theta_t v_x}{v^2} - \frac{\theta^{1+b} u_{xx}}{(1+b)v^2} \right] dx dt \right| \\ &=: |K_1 + K_2 + K_3|. \end{aligned} \quad (69)$$

We will deal with them term by term. For  $K_1$  and  $K_3$ , we have

$$\begin{aligned} |K_1| &= \left| \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right) \left( \frac{2\theta^{1+b} v_x u_x}{(1+b)v^3} \right) dx dt \right| \\ &\leq C \int_0^t \int_0^1 |\theta^{1+2b} \theta_x v_x u_x| \, dx dt \\ &\leq C \left( \max_{(x,t) \in [0,1] \times [0,T]} |u_x| \right) \int_0^t \int_0^1 |\theta^{1+2b} \theta_x v_x| \, dx dt \\ &\leq (C + CZ^{\frac{3}{8}}) \left( \int_0^t \int_0^1 \theta^{1+3b} \theta_x^2 \, dx dt \right)^{\frac{1}{2}} \left( \int_0^t \int_0^1 \theta^{1+b} v_x^2 \, dx dt \right)^{\frac{1}{2}} \\ &\leq (C + CZ^{\frac{3}{8}}) \left[ (C + CY) \int_0^t \max_{x \in [0,1]} \theta^{1+b} \, dt \right]^{\frac{1}{2}} \left[ \left( \sup_{0 \leq t \leq T} \int_0^1 v_x^2 \, dx \right) \left( \int_0^t \max_{x \in [0,1]} \theta^{1+b} \right) \right]^{\frac{1}{2}} \\ &\leq (C + CZ^{\frac{3}{8}}) (C + CY^{\frac{1}{2}}), \end{aligned} \quad (70)$$

and

$$\begin{aligned} |K_3| &= \left| \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right) \left( \frac{\theta^{1+b} u_{xx}}{(1+b)v^2} \right) dx dt \right| \\ &\leq C \left[ \int_0^t \int_0^1 \frac{\theta^b \theta_x^2}{\theta^{1.75}} \theta^{2+2b} \, dx dt \right]^{\frac{1}{2}} \left[ \int_0^t \int_0^1 \theta^{1.75+b} u_{xx}^2 \, dx dt \right]^{\frac{1}{2}} \\ &\leq C \left( \max_{(x,t) \in [0,1] \times [0,T]} \theta^{1+b} \right) \left[ \int_0^t \int_0^1 \frac{\theta^b \theta_x^2}{\theta^{1.75}} \, dx dt \right]^{\frac{1}{2}} \left[ \left( \int_0^t \max_{x \in [0,1]} \theta^{1.75+b} \, dt \right) Z \right]^{\frac{1}{2}} \\ &\leq C(1 + Y^{\frac{1+b}{3+b}}) Z^{\frac{1}{2}}, \end{aligned} \quad (71)$$

For  $K_2$ , we compute

$$\begin{aligned} |K_2| &= \left| \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right) \left( \frac{\theta^b \theta_t v_x}{v^2} \right) dx dt \right| \\ &\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} \, dx dt + C \int_0^t \int_0^1 \frac{\theta^b \theta^{2b} \theta_x^2 v_x^2}{v^2} \, dx dt \\ &\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} \, dx dt + C \max_{(x,t) \in [0,1] \times [0,T]} |\theta^b| \left( \sup_{0 \leq t \leq T} \int_0^1 v_x^2 \, dx \right) \int_0^t \left( \max_{x \in [0,1]} \frac{\theta^{2b} \theta_x^2}{v^2} \right) dt \\ &\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} \, dx dt + C(1 + Y^{\frac{b}{3+b}}) \int_0^t \left( \max_{x \in [0,1]} \frac{\theta^{2b} \theta_x^2}{v^2} \right) dt. \end{aligned} \quad (72)$$

Using  $\theta_x(0, t) = \theta_x(1, t) = 0$ , we have

$$\begin{aligned}
 & \int_0^t \left( \max_{x \in [0,1]} \frac{\theta^{2b} \theta_x^2}{v^2} \right) dt \leq C \int_0^t \int_0^1 \left| \frac{\theta^b \theta_x}{v} \right| \cdot \left| \left( \frac{\theta^b \theta_x}{v} \right)_x \right| dx dt \\
 & \leq C \left( \int_0^t \int_0^1 \frac{\theta^{2b} \theta_x^2}{v^2} dx dt \right)^{\frac{1}{2}} \left( \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right)_x^2 dx dt \right)^{\frac{1}{2}} \\
 & \leq C \left( \max_{(x,t) \in [0,1] \times [0,T]} \theta \right) \left( \int_0^t \int_0^1 \frac{\theta^b \theta_x^2}{\theta^{2-b}} dx dt \right)^{\frac{1}{2}} \left( \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right)_x^2 dx dt \right)^{\frac{1}{2}} \\
 & \leq C(1 + Y^{\frac{1}{3+b}}) \left( \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right)_x^2 dx dt \right)^{\frac{1}{2}}.
 \end{aligned} \tag{73}$$

From the temperature equation (26), we find

$$\begin{aligned}
 & \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right)_x^2 dx dt \\
 & \leq \int_0^t \int_0^1 \theta_t^2 dx dt + \int_0^t \int_0^1 \left[ \left( \frac{u_x^2}{v} \right)^2 + \left( \frac{\theta u_x}{v} \right)^2 \right] dx dt \\
 & \leq \int_0^t \int_0^1 \theta_t^2 dx dt + C \left( \max_{(x,t) \in [0,1] \times [0,T]} |u_x^2| \right) \left( \int_0^t \int_0^1 (u_x^2 + \theta^2) dx dt \right) \\
 & \leq C \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + CZ^{\frac{3}{4}} + C.
 \end{aligned} \tag{74}$$

Therefore, we conclude from (72), (73), and (74), that

$$\begin{aligned}
 |K_2| & \leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C(1 + Y^{\frac{1+b}{3+b}}) [1 + Z^{\frac{3}{4}} + \left( \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt \right)^{\frac{1}{2}}] \\
 & \leq \frac{1}{4} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C(Y^{\frac{2+2b}{3+b}} + Z^{\frac{3}{4}} + 1).
 \end{aligned} \tag{75}$$

We sum (69), (70), (75), and (71) up to obtain

$$\begin{aligned}
 & \int_0^t \int_0^1 \frac{\theta^b \theta_x}{v} \tilde{K} dx dt \\
 & \leq \frac{1}{4} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C(1 + Z^{\frac{3}{8}} Y^{\frac{1}{2}} + Y^{\frac{1+b}{3+b}} Z^{\frac{1}{2}} + Y^{\frac{2+2b}{3+b}} + Z^{\frac{3}{4}}).
 \end{aligned} \tag{76}$$

We now turn to estimate the last term in (67),

$$\begin{aligned}
& \left| \int_0^t \int_0^1 \left( \frac{u_x^2}{v} - \frac{\theta}{v} u_x \right) K_t dx dt \right| \\
&= \left| \int_0^t \int_0^1 \left( \frac{u_x^2}{v} - \frac{\theta}{v} u_x \right) \left( \frac{\theta^b \theta_t}{v} - \frac{\theta^{1+b} u_x}{(1+b)v^2} \right) dx dt \right| \\
&\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \int_0^t \int_0^1 \left[ \theta^b u_x^4 + \theta^{2+b} u_x^2 + \theta^{1+b} |u_x|^3 \right] dx dt \\
&\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \int_0^t \int_0^1 \left( \theta^b u_x^4 + \theta^{2+b} u_x^2 \right) dx dt \\
&\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \left( Z^{\frac{3}{4}} Y^{\frac{b}{3+b}} + Y^{\frac{2+b}{3+b}} \right) \left( \int_0^t \int_0^1 u_x^2 dx dt \right) \\
&\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \left( 1 + Z^{\frac{3}{4}} Y^{\frac{b}{3+b}} + Y^{\frac{2+b}{3+b}} \right)
\end{aligned} \tag{77}$$

We now conclude from (67), (68), (76), and (77) that there is some constant  $0 < C(b) < 1$  for  $b \in [0, 1)$ , it holds that

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_0^1 \theta^{2b} \theta_x^2(x, t) dx + \int_0^T \int_0^1 \theta^b \theta_t^2 dx dt \\
&\leq C + C \left( Z^{\frac{3}{4}} + Y^{\frac{2+2b}{3+b}} + Y^{\frac{2+b}{3+b}} \right) + C \left( Z^{\frac{3}{8}} Y^{\frac{1}{2}} + Y^{\frac{1+b}{3+b}} Z^{\frac{1}{2}} + Z^{\frac{3}{4}} Y^{\frac{b}{3+b}} \right) \\
&\leq C + \frac{1}{2} Y + CZ^{C(b)},
\end{aligned} \tag{78}$$

where we have used Young's inequality, and the following facts

$$\frac{2+2b}{3+b} < 1, \quad \frac{1+b}{3+b} < \frac{1}{2}, \quad \text{and} \quad \frac{b}{3+b} < \frac{1}{4}, \quad \text{for } b \in (0, 1).$$

This gives the proof of Lemma 2.9.  $\square$

We now deal with the case when  $b \geq 1$ .

**Lemma 2.10.** *When  $b \geq 1$ , there exists a constant  $C > 0$  such that*

$$\sup_{0 \leq t \leq T} \int_0^1 u_x^2 dx + \int_0^T \int_0^1 u_{xx}^2 dx dt \leq C, \tag{79}$$

and

$$\int_0^T \int_0^1 \theta_x^2 dx dt \leq C. \tag{80}$$

*Proof.* First of all, we note from Lemmas 2.1–2.5, that

$$\int_0^T \int_0^1 \frac{\theta^b \theta_x^2}{\theta^{1+p}} dx dt \leq C,$$

for any  $p > 0$ . Therefore, if  $b > 1$ , we can choose  $p = b - 1$  to obtain (80).

We rewrite momentum equation in the following form

$$u_t - \frac{u_{xx}}{v} = -\frac{u_x v_x}{v^2} - \frac{\theta_x}{v} + \frac{\theta v_x}{v^2}. \tag{81}$$



Multiplying both side of (81) with  $u_{xx}$ , then integrating in  $x$  over  $[0, 1]$ , one has

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^1 u_x^2 dx + \int_0^1 \frac{u_{xx}^2}{v} dx \\
& \leq \left| \int_0^1 \frac{u_x v_x}{v^2} u_{xx} dx \right| + \left| \int_0^1 \frac{\theta_x}{v} u_{xx} dx \right| + \left| \int_0^1 \frac{\theta v_x}{v^2} u_{xx} dx \right| \\
& \leq \frac{1}{4} \int_0^1 \frac{u_{xx}^2}{v} dx + C \left( \int_0^1 u_x^2 v_x^2 + v_x^2 \theta^2 + \theta_x^2 \right) dx \\
& \leq \frac{1}{4} \int_0^1 \frac{u_{xx}^2}{v} dx + C \left( \max_{x \in [0,1]} (u_x^2 + \theta^2) \right) \left( \sup_{0 \leq t \leq T} \int_0^1 v_x^2 dx \right) + C \int_0^1 \theta_x^2 dx \\
& \leq \frac{1}{4} \int_0^1 \frac{u_{xx}^2}{v} dx + C \left( \max_{x \in [0,1]} (u_x^2 + \theta^2) \right) + C \int_0^1 \theta_x^2 dx.
\end{aligned} \tag{82}$$

We note that

$$\max_{x \in [0,1]} u_x^2 \leq C \left( \left( \int_0^1 u_x^2 dx \right)^{\frac{1}{2}} \left( \int_0^1 u_{xx}^2 dx \right)^{\frac{1}{2}} + \int_0^1 u_x^2 dx \right), \tag{83}$$

which reduces (82) to

$$\begin{aligned}
& \frac{d}{dt} \int_0^1 u_x^2 dx + \int_0^1 \frac{u_{xx}^2}{v} dx \\
& \leq C \left( \max_{x \in [0,1]} \theta^2 \right) + C \int_0^1 u_x^2 dx + C_1 \int_0^1 \theta_x^2 dx.
\end{aligned} \tag{84}$$

Now, if  $b > 1$ , we use (80) and integrate (84) in time to obtain (79).

In the rest of this proof, we will focus on the case when  $b = 1$ . To this end, we define

$$\eta(\theta) = \int_{\sigma}^{\theta} \ln\left(\frac{\xi}{\sigma}\right) d\xi > 0, \tag{85}$$

where

$$0 < \sigma = \frac{1}{2} \min_{(x,t) \in [0,1] \times [0,T]} \theta.$$

Using the temperature equation with  $b = 1$ , we get

$$\begin{aligned}
& \frac{d}{dt} \int_0^1 \eta(\theta) dx + \int_0^1 \frac{\theta_x^2}{v} dx \\
& = \int_0^1 \frac{u_x^2}{v} \ln\left(\frac{\theta}{\sigma}\right) dx - \int_0^1 \frac{\theta u_x}{v} \ln\left(\frac{\theta}{\sigma}\right) dx \\
& \leq C \left( \max_{x \in [0,1]} \theta + 1 \right) \int_0^1 u_x^2 dx + \int_0^1 \theta^{2.5} dx.
\end{aligned} \tag{86}$$

Choosing a positive number  $M$  large enough such that

$$\frac{M}{v} > 2C_1$$

where  $C_1$  is given in (84). We compute  $M \times (86) + (84)$  to get

$$\begin{aligned}
& \frac{d}{dt} \int_0^1 (u_x^2 + M\eta(\theta)) dx + \int_0^1 \left[ \left( \frac{M}{v} - C_1 \right) \theta_x^2 + \frac{u_{xx}^2}{v} \right] dx \\
& \leq C(M) \left( \max_{x \in [0,1]} \theta + 1 \right) \int_0^1 u_x^2 dx + M \int_0^1 \theta^{2.5} dx + C \left( \max_{x \in [0,1]} \theta^2 \right),
\end{aligned} \tag{87}$$

With the help of Gronwall inequality, we now integrate (87) in time to conclude this lemma for the case when  $b = 1$ . This completes the proof.  $\square$

In order to control the difficulty appears in heat conductivity when  $b \geq 1$ , we now improve the estimates on  $\theta$ . The following lemma gives estimate on the  $L^p$  norm of  $\theta$  for any  $1 < p < \infty$ .

**Lemma 2.11.** *When  $b \geq 1$ , for any  $p > 1$ , there exists a constant  $C$  such that*

$$\sup_{0 \leq t \leq T} \int_0^1 \theta^p dx + \int_0^T \int_0^1 \theta^{p+b-2} \theta_x^2 dx dt + \int_0^T \max_{x \in [0,1]} u_x^2 dt \leq C, \tag{88}$$

*Proof.* Multiply  $\theta^{p-1}$  for  $p > 1$  on both side of the temperature equation (26), we get

$$\begin{aligned}
& \frac{1}{p} \frac{d}{dt} \int_0^1 \theta^p dx + \int_0^1 \frac{(p-1)\theta^{p+b-2} \theta_x^2}{v} dx \\
& = \int_0^1 \frac{\theta^{p-1} u_x^2}{v} dx - \int_0^1 \frac{\theta^p u_x}{v} dx \\
& \leq \max_{x \in [0,1]} |u_x|^2 \int_0^1 \theta^{p-1} dx + \max_{x \in [0,1]} |u_x| \int_0^1 \theta^p dx \\
& \leq C \max_{x \in [0,1]} (1 + |u_x|^2) \int_0^1 \theta^p dx
\end{aligned} \tag{89}$$

We recall that

$$\max_{x \in [0,1]} (|u_x|^2) \leq C \int_0^1 u_x^2 dx + C \int_0^1 u_{xx}^2 dx,$$

and so Lemma 2.10 implies

$$\int_0^T \max_{x \in [0,1]} u_x^2 dt \leq C.$$

Therefore, the Gronwall inequality and (89) imply (88).  $\square$

The following lemma is an improved version of Lemma 2.8 when  $b \geq 1$ .

**Lemma 2.12.** *When  $b \geq 1$ , for any positive number  $p$  there exists a constant  $C$  such that*

$$\max_{(x,t) \in [0,1] \times [0,T]} \theta \leq C + CY^{\frac{1}{3+b+p}} \tag{90}$$

*Proof.* Based on (88), we have

$$\begin{aligned}
\max_{(x,t) \in [0,1] \times [0,T]} \theta^{3+b+p} &= C + \max_{(x,t) \in [0,1] \times [0,T]} (\theta^{\frac{3+b+p}{2}} - \theta(a(t), t)^{\frac{3+b+p}{2}})^2 \\
&\leq C + \sup_{0 \leq t \leq T} \left( \int_0^1 \theta^{\frac{1+b+p}{2}} \theta_x dx \right)^2 \\
&\leq C + \sup_{0 \leq t \leq T} \left( \int_0^1 \theta^{2b} \theta_x^2 dx \right) \left( \int_0^1 \theta^{p+1-b} dx \right) \\
&\leq C + CY
\end{aligned} \tag{91}$$

Then (90) is proved.  $\square$

We are now ready to prove the following key lemma which is stronger than Lemma 2.8 due to strong results obtained in Lemmas 2.9–2.12.

**Lemma 2.13.** *When  $b \geq 1$  there exists a constant  $C > 0$  such that*

$$\int_0^T \int_0^1 \theta^b \theta_t^2 dx dt + \sup_{0 \leq t \leq T} \int_0^1 \theta^{2b} \theta_x^2 dx \leq C. \tag{92}$$

*Proof.* Recall the following key equation (67) for the proof of this lemma

$$\begin{aligned}
&\frac{1}{2} \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right)^2 dx + \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt \\
&= C + \int_0^t \int_0^1 \frac{\theta^{1+b} \theta_t u_x}{(1+b)v^2} dx dt - \int_0^t \int_0^1 \frac{\theta^b \theta_x}{v} \tilde{K} dx dt + \int_0^t \int_0^1 \left( \frac{u_x^2}{v} - \frac{\theta}{v} u_x \right) K_t dx dt.
\end{aligned} \tag{93}$$

We will need to estimate last three terms in (93) again under the assumption  $b \geq 1$ . The first term can be treated as following

$$\begin{aligned}
\left| \int_0^t \int_0^1 \frac{\theta^{1+b} \theta_t u_x}{(1+b)v^2} dx dt \right| &\leq \frac{1}{4} \int_0^T \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \int_0^T \int_0^1 \theta^{2+b} u_x^2 dx dt \\
&\leq \frac{1}{4} \int_0^T \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \int_0^t \max_{(x,t) \in [0,1] \times [0,T]} |u_x|^2 dt \\
&\leq C + \frac{1}{4} \int_0^T \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt.
\end{aligned} \tag{94}$$

The second term reads

$$\begin{aligned}
\left| \int_0^t \int_0^1 \frac{\theta^b \theta_x}{v} \tilde{K} dx dt \right| &= \left| \int_0^t \int_0^1 \frac{\theta^b \theta_x}{v} \left[ \frac{2\theta^{1+b} v_x u_x}{(1+b)v^3} - \frac{\theta^b \theta_t v_x}{v^2} - \frac{\theta^{1+b} u_{xx}}{(1+b)v^2} \right] dx dt \right| \\
&=: |K_1 + K_2 + K_3|.
\end{aligned} \tag{95}$$

We will deal with them term by term. For  $K_1$  and  $K_3$ , we have

$$\begin{aligned}
|K_1| &= \left| \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right) \left( \frac{2\theta^{1+b} v_x u_x}{(1+b)v^3} \right) dx dt \right| \\
&\leq C \int_0^t \int_0^1 \theta^{2+4b} \theta_x^2 dx dt + C \int_0^t \int_0^1 u_x^2 v_x^2 dx dt \\
&\leq C + C \int_0^t \max_{x \in [0,1]} u_x^2 dt \\
&\leq C,
\end{aligned} \tag{96}$$

and

$$\begin{aligned}
|K_3| &= \left| \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right) \left( \frac{\theta^{1+b} u_{xx}}{(1+b)v^2} \right) dx dt \right| \\
&\leq C \int_0^T \int_0^1 \theta^{2+4b} \theta_x^2 dx dt + \int_0^T \int_0^1 u_{xx}^2 dx dt \\
&\leq C.
\end{aligned} \tag{97}$$

For  $K_2$ , we compute

$$\begin{aligned}
|K_2| &= \left| \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right) \left( \frac{\theta^b \theta_t v_x}{v^2} \right) dx dt \right| \\
&\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \int_0^t \int_0^1 \frac{\theta^b \theta^{2b} \theta_x^2 v_x^2}{v^2} dx dt \\
&\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \max_{(x,t) \in [0,1] \times [0,T]} |\theta^b| \left( \sup_{0 \leq t \leq T} \int_0^1 v_x^2 dx \right) \int_0^t \left( \max_{x \in [0,1]} \frac{\theta^{2b} \theta_x^2}{v^2} \right) dt \\
&\leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C(1 + Y^{\frac{b}{3+b+\tilde{p}}}) \int_0^t \left( \max_{x \in [0,1]} \frac{\theta^{2b} \theta_x^2}{v^2} \right) dt,
\end{aligned} \tag{98}$$

where,  $\tilde{p} > 0$  will be determined later. Using  $\theta_x(0, t) = \theta_x(1, t) = 0$ , we have

$$\begin{aligned}
\int_0^t \left( \max_{x \in [0,1]} \frac{\theta^{2b} \theta_x^2}{v^2} \right) dt &\leq C \int_0^t \int_0^1 \left| \frac{\theta^b \theta_x}{v} \right| \cdot \left| \left( \frac{\theta^b \theta_x}{v} \right)_x \right| dx dt \\
&\leq C \left( \int_0^t \int_0^1 \frac{\theta^{2b} \theta_x^2}{v^2} dx dt \right)^{\frac{1}{2}} \left( \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right)_x^2 dx dt \right)^{\frac{1}{2}} \\
&\leq C \left( \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right)_x^2 dx dt \right)^{\frac{1}{2}}.
\end{aligned} \tag{99}$$

From the temperature equation (26), we find

$$\begin{aligned}
& \int_0^t \int_0^1 \left( \frac{\theta^b \theta_x}{v} \right)_x^2 dx dt \\
& \leq \int_0^t \int_0^1 \theta_t^2 dx dt + \int_0^t \int_0^1 \left[ \left( \frac{u_x^2}{v} \right)^2 + \left( \frac{\theta u_x}{v} \right)^2 \right] dx dt \\
& \leq \int_0^t \int_0^1 \theta_t^2 dx dt + C \int_0^t (\max_{x \in [0,1]} |u_x^2|) \left( \int_0^1 (u_x^2 + \theta^2) dx \right) dt \\
& \leq C \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C.
\end{aligned} \tag{100}$$

Therefore, we conclude from (98), (99), and (100), that

$$\begin{aligned}
|K_2| & \leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C(1 + Y^{\frac{b}{3+b+\bar{p}}}) (C + (\int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt)^{\frac{1}{2}}) \\
& \leq \frac{1}{4} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + CY^{\frac{2b}{3+b+\bar{p}}} + C.
\end{aligned} \tag{101}$$

We sum (95), (96), (101), and (97) up to obtain

$$\int_0^t \int_0^1 \frac{\theta^b \theta_x}{v} \tilde{K} dx dt \leq C + CY^{\frac{2b}{3+b+\bar{p}}} + \frac{1}{4} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt. \tag{102}$$

We now turn to estimate the last term in (93),

$$\begin{aligned}
& \left| \int_0^t \int_0^1 \left( \frac{u_x^2}{v} - \frac{\theta}{v} u_x \right) K_t dx dt \right| \\
& = \left| \int_0^t \int_0^1 \left( \frac{u_x^2}{v} - \frac{\theta}{v} u_x \right) \left( \frac{\theta^b \theta_t}{v} - \frac{\theta^{1+b} u_x}{(1+b)v^2} \right) dx dt \right| \\
& \leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \int_0^t \int_0^1 \left[ \theta^b u_x^4 + \theta^{2+b} u_x^2 + \theta^{1+b} |u_x|^3 \right] dx dt \\
& \leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C \int_0^t \int_0^1 \left( \theta^b u_x^4 + \theta^{2+b} u_x^2 \right) dx dt \\
& \leq \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + C(1 + Y^{\frac{b}{3+b+\bar{p}}}) \left( \int_0^t \max_{x \in [0,1]} (u_x^2 + \theta^2) dt \right) \left( \sup_{0 \leq t \leq T} \int_0^1 u_x^2 dx \right) \\
& \leq C + \frac{1}{8} \int_0^t \int_0^1 \frac{\theta^b \theta_t^2}{v} dx dt + CY^{\frac{b}{3+b+\bar{p}}}.
\end{aligned} \tag{103}$$

We now conclude from (93), (94), (102), and (103) that

$$\begin{aligned}
\sup_{0 \leq t \leq T} \int_0^1 \theta^{2b} \theta_x^2 dx + \int_0^T \int_0^1 \theta^b \theta_t^2 dx dt & \leq C + CY^{\frac{b}{3+b+\bar{p}}} + CY^{\frac{2b}{3+b+\bar{p}}} \\
& \leq C + \frac{1}{2} Y,
\end{aligned} \tag{104}$$

where we have chosen  $\tilde{p} = b$  and applied Young's inequality. We thus conclude from (104) that

$$\sup_{0 \leq t \leq T} \int_0^1 \theta^{2b} \theta_x^2 dx + \int_0^T \int_0^1 \theta^b \theta_t^2 dx dt \leq C.$$

which completes the proof.  $\square$

Finally, we are ready to give the estimate on  $Z$  and conclude the estimate (8) in Theorem 1.1.

**Lemma 2.14.** *There exists a constant  $C > 0$  such that*

$$\sup_{0 \leq t \leq T} \int_0^1 u_t^2 dx + \sup_{0 \leq t \leq T} \int_0^1 u_{xx}^2 dx + \int_0^T \int_0^1 u_{xt}^2 dx dt \leq C, \quad (105)$$

and

$$\sup_{0 \leq t \leq T} \int_0^1 \theta_t^2 dx + \sup_{0 \leq t \leq T} \int_0^1 \theta_{xx}^2 dx + \int_0^T \int_0^1 \theta_{xt}^2 dx dt \leq C. \quad (106)$$

*Proof.* Differentiating momentum equation with respect to  $t$ , we have

$$u_{tt} = \left(\frac{u_x}{v}\right)_{xt} - \left(\frac{\theta}{v}\right)_{xt}. \quad (107)$$

Multiplying it with  $u_t$ , and integrating it over  $[0, 1]$ , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 u_t^2 dx + \int_0^1 \frac{u_{xt}^2}{v} dx \\ &= \int_0^1 \left( \frac{u_x^2}{v^2} + \frac{\theta_t}{v} - \frac{\theta u_x}{v^2} \right) u_{xt} dx \\ &\leq \frac{1}{2} \int_0^1 \frac{u_{xt}^2}{v} dx + C \int_0^1 (u_x^4 + \theta_t^2 + u_x^2 \theta^2) dx \\ &\leq \frac{1}{2} \int_0^1 \frac{u_{xt}^2}{v} dx + C(1 + Z^{\frac{3}{4}}) \int_0^1 (u_x^2 + \theta^2) dx + C \int_0^1 \theta^b \theta_t^2 dx. \end{aligned} \quad (108)$$

With the help of Lemma 2.9 (for  $b < 1$ ) and Lemma 2.12 (for  $b \geq 1$ ), we integrate (108) in time to obtain

$$\begin{aligned} \sup_{0 \leq t \leq T} \int_0^1 u_t^2 dx + \int_0^t \int_0^1 u_{xt}^2 dx dt &\leq C + CZ^{\frac{3}{4}} + CZ^{C(b)} \\ &\leq C + CZ^{C(b)}. \end{aligned} \quad (109)$$

On the other hand, we rewrite the momentum equation as

$$\frac{u_{xx}}{v} = u_t - \frac{u_x v_x}{v^2} - \left(\frac{\theta}{v}\right)_x \quad (110)$$

which implies that

$$\begin{aligned}
Z &\leq C \sup_{0 \leq t \leq T} \left( \int_0^1 u_t^2 dx + \int_0^1 u_x^2 v_x^2 dx + \int_0^1 \theta_x^2 dx + \int_0^1 \theta^2 v_x^2 dx \right) \\
&\leq C(1 + Z^{C(b)} + Y + \max_{(x,t) \in [0,1] \times [0,T]} (u_x^2 + \theta^2)) \\
&\leq C(1 + Z^{C(b)} + Y + Z^{\frac{3}{4}} + Y^{\frac{2}{3+b}}) \\
&\leq C + CZ^{C(b)},
\end{aligned} \tag{111}$$

Since  $0 < C(b) < 1$ , with the help of Young's inequality, we obtain

$$Z \leq C.$$

Therefore, (105) follows.

It remains to prove the second estimate in Lemma 2.14. Differentiating the temperature equation (26) with respect to  $t$ , we have

$$\theta_{tt} = \left( \frac{\theta^b \theta_x}{v} \right)_{xt} + \left( \frac{u_x^2}{v} \right)_t - \left( \frac{\theta u_x}{v} \right)_t. \tag{112}$$

Multiplying (112) with  $\theta_t$ , and integrating in  $x$  over  $[0, 1]$ , after integration by parts, it gives

$$\begin{aligned}
&\frac{d}{dt} \int_0^1 \theta_t^2 dx + \int_0^1 \theta_{xt}^2 dx \\
&\leq C \int_0^1 |\theta_{xt}| (|\theta_x \theta_t| + |\theta_x u_x|) dx + C \int_0^1 |\theta_t| (|u_{xt} u_x| + |u_x^3| + |u_x \theta_t| + |u_{xt}|) dx \\
&\leq \frac{1}{2} \int_0^1 \theta_{xt}^2 dx + C \int_0^1 (1 + \theta_x^2 + |u_x|^2) \theta_t^2 dx + C \int_0^1 (u_x^2 + \theta_x^2 + u_{xt}^2) dx \\
&\leq \frac{1}{2} \int_0^1 \theta_{xt}^2 dx + C (\max_{x \in [0,1]} \theta_x^2) \int_0^1 \theta_t^2 dx + C + C \int_0^1 (\theta_t^2 + u_{xt}^2) dx,
\end{aligned} \tag{113}$$

which implies that

$$\begin{aligned}
&\frac{d}{dt} \int_0^1 \theta_t^2 dx + \int_0^1 \theta_{xt}^2 dx \\
&\leq C + C (\max_{x \in [0,1]} \theta_x^2) \int_0^1 \theta_t^2 dx + C \int_0^1 (\theta_t^2 + u_{xt}^2) dx.
\end{aligned} \tag{114}$$

We note from (99) and (100) that

$$\int_0^t (\max_{x \in [0,1]} \theta_x^2) dt \leq C \int_0^t (\max_{x \in [0,1]} \frac{\theta^{2b} \theta_x^2}{v^2}) dt \leq C.$$

Therefore, we apply Gronwall inequality to (114) to obtain

$$\sup_{0 \leq t \leq T} \int_0^1 \theta_t^2 dx + \int_0^T \int_0^1 \theta_{xt}^2 dx dt \leq C. \tag{115}$$

Finally, we rewrite the temperature equation (26)

$$\frac{\theta^b \theta_{xx}}{v} = \theta_t + \frac{\theta}{v} u_x - \frac{u_x^2}{v} - \frac{b \theta^{b-1} \theta_x^2}{v} + \frac{\theta^b \theta_x v_x}{v^2}, \tag{116}$$

which gives

$$\begin{aligned}
\int_0^1 \theta_{xx}^2 dx &\leq C \int_0^1 (\theta_t^2 + u_x^2 + u_x^4 + \theta_x^4 + \theta_x^2 v_x^2) dx \\
&\leq C + C \max_{x \in [0,1]} \theta_x^2 \\
&\leq C + C \int_0^1 |\theta_x \theta_{xx}| dx \\
&\leq C + C \left( \int_0^1 \theta_{xx}^2 dx \right)^{\frac{1}{2}},
\end{aligned} \tag{117}$$

which implies

$$\sup_{0 \leq t \leq T} \int_0^1 \theta_{xx}^2 dx \leq C.$$

We thus complete the proof of this lemma.  $\square$

It is clear that we have carried out all estimates in (8) of Theorem 1.1. Theorem 1.1 then follows by standard procedures. We omit the details.

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(R. Pan) SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA 30332, U.S.A.

*E-mail address:* panrh@math.gatech.edu

(W. Zhang) SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA 30332, U.S.A.

*E-mail address:* wzhang@math.gatech.edu