

INITIAL BOUNDARY VALUE PROBLEM FOR 2D BOUSSINESQ EQUATIONS WITH TEMPERATURE-DEPENDENT HEAT DIFFUSION

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ABSTRACT. We consider the initial-boundary value problem of two-dimensional inviscid heat conductive Boussinesq equations with nonlinear heat diffusion over a bounded domain with smooth boundary. Under slip boundary condition of velocity and the homogeneous Dirichlet boundary condition for temperature, we show that there exists a unique global smooth solution to the initial-boundary value problem for H^3 initial data. Moreover, we will show that the temperature converges exponentially to zero as time goes to infinity, and the velocity and vorticity are uniformly bounded in time.

1. INTRODUCTION

In this paper, we consider the 2D inviscid heat conductive Boussinesq equations

$$\begin{cases} u_t + u \cdot \nabla u + \nabla P = \theta \mathbf{e}_2, \\ \theta_t + u \cdot \nabla \theta = \nabla \cdot (\kappa(\theta) \nabla \theta), \\ \nabla \cdot u = 0, \end{cases} \quad (1)$$

with the initial-boundary conditions

$$\begin{cases} (u, \theta)(x, 0) = (u_0, \theta_0)(x), & x \in \Omega, \\ u \cdot \mathbf{n}|_{\partial\Omega} = 0, & \theta|_{\partial\Omega} = 0, \end{cases} \quad (2)$$

where $u = (u_1, u_2)$, P and θ are the velocity vector field, the pressure, and (modified) temperature, respectively. Ω is a bounded domain in \mathbb{R}^2 with smooth boundary $\partial\Omega$, and \mathbf{n} is the unit outward normal vector of $\partial\Omega$. $\mathbf{e}_2 = (0, 1)^T$ is the unit direction of gravity. κ models thermal diffusion, and is a smooth function of τ satisfying

$$\kappa(\tau) \geq \underline{\kappa} > 0, \quad (3)$$

for a positive constant $\underline{\kappa}$ and for any $\tau \in \mathbb{R}$. We remark that this condition can be relaxed to $\kappa(\tau) > 0$, which is equivalent to (3) due to the maximum principle in Lemma 2.7 below.

Boussinesq system arises from the description of natural convection, providing an accurate approximation to 3D incompressible fluids in many important applications. In more recent applications, it models geophysical flows such as atmospheric fronts and ocean circulations (see for example [18, 14]). In fluid mechanics, system (1) is used in the field of buoyancy-driven flow. It describes the motion of an inviscid incompressible fluid subject to convective heat transfer under the influence of gravitational force (cf. [15]). On the other hand, the significant mathematical interest of such system roots from the role of a simplified model of 3D incompressible fluids sharing the same vortex stretching effects. It is believed that the understanding of the behavior of Boussinesq type system will shed the light to the global regularity problem of 3D incompressible fluids. There are a great amount of literatures concerning partial answers to this question. For more information on 3D incompressible fluids, we refer the reader to [19, 11] and the references therein for detailed discussions.

While the global regularity problem for 2D Boussinesq system in the absence of any dissipation still remains as an outstanding open problem, many significant results were achieved in recent years when dissipations present. The general Boussinesq equations take the following form

$$\begin{cases} u_t + u \cdot \nabla u + \nabla P = \nabla \cdot (\mu \nabla u) + \theta \mathbf{e}_2, \\ \theta_t + u \cdot \theta = \nabla \cdot (\kappa \nabla \theta), \\ \nabla \cdot u = 0 \end{cases} \quad (4)$$

When both μ and κ are positive constants, the problem was solved in has been analyzed in great extent (see for instance [2, 16, 17]). In recent years, significant attention has been paid to the cases of partial dissipation where μ and κ are constants and either $\mu > 0$, $\kappa = 0$, or $\mu = 0$, $\kappa > 0$. For the results in this direction, we refer the readers to [3, 5] for the global regularity problem to the Cauchy problem, and to [9, 22] for the initial-boundary value problem.

In some realistic applications such as some plasma flows and mantle convection in geophysical fluid dynamics, the viscosity μ and thermal conductivity κ are sensitive to the variation of temperature and they are usually functions of θ . When κ and μ depend on θ , only a few results are viable, see [12, 13, 21]. The nonlinearity of temperature-dependent dissipation introduced some major difficulty in analysis comparing to the case when μ and κ are constants. Roughly speaking, neat elliptic estimate in the case of constant μ and κ has to be replaced by the much more complicated elliptic theory with variable dependent coefficients. Furthermore, since θ is also the unknown function, it makes the case even more delicate. In [12, 13], Lorca and Boldrini proved that the global existence of weak solution with small initial data and the local existence of strong solution for general data to the problem (4). Recently, Wang and Zhang [21] obtain the global existence of smooth solution to the Boussinesq system (4) with full dissipations (both μ and κ have positive lower bound), see also [20] for initial boundary value problem. The global regularity problem for partial dissipation cases is open. In this paper, setting $\mu = 0$ and $\kappa(\theta) > 0$, we will make the first attempt to tackle this problem, i.e., the global regularity problem of the initial-boundary value problem to the system (1) in a bounded smooth domain.

As usual, for global existence of smooth solutions, we require the following compatibility conditions:

$$\begin{aligned} u_0 \cdot \mathbf{n}|_{\partial\Omega} &= 0, & \nabla \cdot u_0 &= 0, \\ \theta_0|_{\partial\Omega} &= 0, & u_0 \cdot \nabla\theta_0 - \nabla \cdot (\kappa(\theta_0)\nabla\theta_0)|_{\partial\Omega} &= 0. \end{aligned} \tag{5}$$

The main result of this paper is stated in the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. If $(u_0(x), \theta_0(x)) \in H^3(\Omega)$ satisfies the compatibility conditions (5), then there exists a unique solution (u, θ) to (1)-(2) globally in time such that*

$$u \in C([0, T]; H^3(\Omega)), \quad \theta \in C([0, T]; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega))$$

for any $T > 0$.

Moreover, there exist positive constants α, β, C independent of t such that, for any fixed $p \in [2, \infty)$, it holds that

$$\begin{aligned} \|\theta(\cdot, t)\|_{H^3(\Omega)} &\leq \alpha e^{-\beta t}, \quad \|\theta_t\|_{L^2(0,t;H^2(\Omega))} \leq C \quad \forall t \geq 0, \\ \|u(\cdot, t)\|_{W^{1,p}(\Omega)} &\leq C, \quad \|\omega(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \forall t \geq 0, \end{aligned}$$

where $\omega = u_{2x} - u_{1y}$ is the 2D vorticity.

The proof of Theorem 1.1 consists of two main parts. First, one shows the global existence of weak solutions to (1)-(2), that is, solutions satisfying the following definition:

Definition 1.1. *The pair (u, θ) is called a global weak solution to (1)-(2) if for any $T > 0$, $u \in C([0, T]; H^1(\Omega))$, $\theta \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, and it holds that*

$$\begin{aligned} \int_{\Omega} u_0 \cdot \varphi(x, 0) d\mathbf{x} + \int_0^T \int_{\Omega} (u \cdot \varphi_t + u \cdot (u \cdot \nabla \varphi) + \theta \mathbf{e}_2 \cdot \varphi) d\mathbf{x} dt &= 0, \\ \int_{\Omega} \theta_0 \psi(x, 0) d\mathbf{x} + \int_0^T \int_{\Omega} (\theta \psi_t + \theta u \cdot \nabla \psi - \kappa(\theta) \nabla \theta \cdot \nabla \psi) d\mathbf{x} dt &= 0 \end{aligned}$$

for any $\varphi = (\varphi_1, \varphi_2) \in C^\infty(\Omega \times [0, T])^2$ satisfying $\varphi(x, T) = 0$, $\nabla \cdot \varphi = 0$, and $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$ and for any $\psi \in C_0^\infty(\Omega \times [0, T])$ satisfying $\psi(x, T) = 0$.

The existence of such a global weak solution follows a standard fixed point argument, c.f. [12, 13]. For a detailed argument using Schauder fixed point argument, we refer the readers to [10]. We omit the details for this part.

We will then focus on the proof of global regularity and uniqueness of solutions in this paper. Rather than the techniques adopted by [21, 20] where full dissipation presents, we will use a relatively simple method basing mainly on the decay of temperature and basic energy estimates to achieve our goals. More precisely, we will first build up decay estimate on temperature, then perform energy estimates on velocity using elliptic theory. The regularity estimates on temperature and velocity enables the decay estimates of temperature and then energy estimate of velocity at the next level. This procedure was repeated up to H^3 , and thus claims the proof of Theorem 1.1.

The rest of paper consists of three sections. In section 2, we describe the basic facts to be used throughout the paper; we also give the global existence of weak solution. Section 3 is devoted to prove the global existence and regularity of the solution in Theorem 1.1. Finally, in section 4, we finish the proof of the uniqueness in Theorem 1.1. Through this paper, we will be using some generic constants. Unless specified, c_i ($i = 0, \dots, 32$) denote generic constants which are independent of time t ; while C is a positive constant which depends on c_i , Ω , α and β .

2. PRELIMINARIES

In order to prove the Theorem 1.1, we need the following lemmas. First, we need the following Sobolev Embeddings and Ladyzhenskaya's inequalities, which are well-know and standard, (see [8]).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Then*

- (1) $\|f\|_{L^\infty(\Omega)}^2 \leq c_1 \|f\|_{W^{1,p}(\Omega)}^2$ for all $p > 2$;
- (2) $\|f\|_{L^p(\Omega)}^2 \leq c_2 \|f\|_{H^1(\Omega)}^2$ for all $1 \leq p < \infty$;
- (3) $\|f\|_{L^4(\Omega)}^2 \leq c_3 \left(\|f\|_{L^2(\Omega)} \|\nabla f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}^2 \right)$, $\forall f : \Omega \rightarrow \mathbb{R}$ and $f \in H^1(\Omega)$;
- (4) $\|f\|_{L^4(\Omega)}^2 \leq c_4 \|f\|_{L^2(\Omega)} \|\nabla f\|_{L^2(\Omega)}$, $\forall f : \Omega \rightarrow \mathbb{R}$ and $f \in H_0^1(\Omega)$.

Next, we recall some classical result on elliptic equations (cf. [4]).

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Consider the Dirichlet problem:*

$$\begin{cases} \Delta\theta = f & \text{in } \Omega, \\ \theta = 0 & \text{on } \partial\Omega. \end{cases}$$

If $f \in W^{m,p}$, then $\theta \in W^{m+2,p}$ and there exists a constants c_5 depending only on p , m and Ω such that

$$\|\theta\|_{W^{m+2,p}} \leq c_5 \|f\|_{W^{m,p}},$$

for any $p \in (1, \infty)$ and the integer $m \geq -1$.

The next two lemmas are useful in the estimation of the velocity field.

Lemma 2.3. ^[1] *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$, and let $u \in W^{s,p}(\Omega)$ be a vector-valued function satisfying $\nabla \cdot u = 0$ and $u \cdot \mathbf{n}|_{\partial\Omega} = 0$, where \mathbf{n} is the unit outward normal to $\partial\Omega$. Then there exists a constant $c_6 = c_6(s, p, \Omega)$ such that*

$$\|u\|_{W^{s,p}} \leq c_6 (\|\nabla \times u\|_{W^{s-1,p}} + \|u\|_{L^p})$$

for any $s \geq 1$ and $p \in (1, \infty)$.

Lemma 2.4. ^[7] *Assume $f, g \in H^N(\mathbb{R}^n)$. Then for any multi-index $\eta = (\eta_1, \dots, \eta_n)$, $|\eta| \leq N$, it holds that*

$$\|\nabla^\eta(fg)\|_{L^2} \leq c_7 (\|f\|_{L^\infty} \|\nabla^N g\|_{L^2} + \|\nabla^N f\|_{L^2} \|g\|_{L^\infty}),$$

for some $c_7 = c_7(N)$.

Lemma 2.5. ^[6] *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Consider the initial-boundary value problem (IBVP):*

$$\begin{cases} u_t + u \cdot \nabla u + \nabla P = G, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad u \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \quad (6)$$

where \mathbf{n} is the unit outward normal to $\partial\Omega$. Let $u_0(x) \in C^{1+\gamma}(\bar{\Omega})$ satisfying $\nabla \cdot u_0(x) = 0$, $u_0(x) \cdot \mathbf{n}|_{\partial\Omega} = 0$. For any fixed $T > 0$, let $G \in C([0, T]; C^{1+\gamma}(\bar{\Omega}))$ for some $0 < \gamma < 1$. Then there exists a solution (u, P) to IBVP (6) such $(u, P) \in C^1(\bar{\Omega} \times [0, T])$.

As explained in the introduction, the global existence of weak solutions of (1)-(2) can be obtained by a fixed point argument and the method of energy estimates.

Lemma 2.6. *Under the assumptions of Theorem 1.1, there exists a global weak solution (u, θ) to (1)-(2), as defined in Definition 1.1, such that, for any $T > 0$, $u \in C([0, T]; H^1(\Omega))$ and $\theta \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$.*

The proof is standard and we omit the details of the proof. For more details, we refer readers to [11] and to [10].

The next lemma gives a maximum principle for the solution of (1)-(2).

Lemma 2.7. *Let (u, θ) be a weak solution of (1)-(2), then*

$$\|\theta\|_{L^\infty(\Omega \times [0, t])} \leq \|\theta_0\|_{L^\infty(\Omega)} \quad \forall t \geq 0.$$

The proof of this lemma is standard, we refer readers to [13] for more details. As an immediately consequence of Lemma 2.7, one has

$$\kappa(\theta) \leq \bar{\kappa} := \sup_{\|\theta\| \leq \|\theta_0\|_{L^\infty(\Omega)}} \kappa(\theta).$$

In order to prove Theorem 1.1, it is convenient to introduce a new quantity

$$\hat{\theta} = \int_0^\theta \kappa(z) dz, \tag{7}$$

which satisfies the following equation

$$\kappa(\theta) \Delta \hat{\theta} = \hat{\theta}_t + u \cdot \nabla \hat{\theta}, \tag{8}$$

with the following initial-boundary conditions

$$\begin{cases} \hat{\theta}(x, 0) = \int_0^{\theta_0(x)} \kappa(z) dz := \hat{\theta}_0(x) & \text{in } \Omega, \\ \hat{\theta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

After some simple calculations, it is clear that

Lemma 2.8. *Let $\hat{\theta}$ be defined in (7), it holds that*

$$\begin{aligned} \|\nabla \hat{\theta}\|_{L^2} &\leq \bar{\kappa} \|\nabla \theta\|_{L^2}, & \|\hat{\theta}_t\|_{L^2} &\leq \bar{\kappa} \|\theta_t\|_{L^2}, \\ \|\nabla \theta\|_{L^2} &\leq \underline{\kappa}^{-1} \|\nabla \hat{\theta}\|_{L^2}, & \|\theta_t\|_{L^2} &\leq \underline{\kappa}^{-1} \|\hat{\theta}_t\|_{L^2}. \end{aligned}$$

3. GLOBAL REGULARITY AND LONG TIME BEHAVIOR

In this section, we shall establish the regularity and uniqueness of the solution obtained in Lemma 2.6, and therefore give a proof of our main result, Theorem 1.1. The following theorem gives the key estimates.

Theorem 3.1. *Under the assumptions of Theorem 1.1, the solution obtained in Lemma 2.6 satisfies*

$$u \in C([0, T]; H^3(\Omega)), \quad \theta \in C([0, T]; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega))$$

for any $T > 0$.

Moreover, there exist positive constants α, β, C independent of t such that for any fixed $p \in [2, \infty)$, it holds that

$$\begin{aligned} \|\theta(\cdot, t)\|_{H^3(\Omega)} &\leq \alpha e^{-\beta t}, \quad \|\theta_t\|_{L^2(0, t; H^2(\Omega))} \leq C \quad \forall t \geq 0, \\ \|u(\cdot, t)\|_{W^{1, p}(\Omega)} &\leq C, \quad \|\omega(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \forall t \geq 0, \end{aligned}$$

where $\omega = u_{2x} - u_{1y}$ is the 2D vorticity.

The proof of Theorem 3.1 is based on several steps of careful energy estimates which are stated as a sequence of lemmas. First, we give the decay estimate of $\|\theta\|_{L^2}$.

Lemma 3.1. *Under the assumptions of Theorem 1.1, it holds that*

$$\|\theta(\cdot, t)\|_{L^2}^2 \leq \|\theta_0\|_{L^2}^2 e^{-2\beta t}, \quad \int_0^t e^{\beta\tau} \|\nabla\theta(\cdot, \tau)\|_{L^2}^2 d\tau \leq \alpha_0 \|\theta_0\|_{L^2}^2 \quad \forall t \geq 0,$$

where $\beta = \underline{\kappa} c_0^{-1}$ with c_0 the constant in Poincaré inequality on the domain Ω , and $\alpha_0 = \underline{\kappa}^{-1}$.

Proof. Multiplying (1)₂ with 2θ , and then integrating over Ω , we have

$$\frac{d}{dt} \|\theta\|_{L^2}^2 + 2\underline{\kappa} \|\nabla\theta\|_{L^2}^2 \leq 0. \quad (10)$$

Since $\theta|_{\partial\Omega} = 0$, Poincaré inequality implies that

$$\|\theta\|_{L^2}^2 \leq c_0 \|\nabla\theta\|_{L^2}^2 \quad (11)$$

for some constant c_0 depending only on Ω . Using (11) we update (10) as

$$\frac{d}{dt} \|\theta\|_{L^2}^2 + \frac{2\underline{\kappa}}{c_0} \|\theta\|_{L^2}^2 \leq 0,$$

which yields immediately that

$$\|\theta(\cdot, t)\|_{L^2}^2 \leq \|\theta_0\|_{L^2}^2 e^{-2\beta t} \quad \forall t \geq 0, \quad (12)$$

where $\beta = \underline{\kappa}c_0^{-1}$.

Then, we multiply (10) by $e^{\beta t}$ and use (12) to get

$$\frac{d}{dt} \left(e^{\beta t} \|\theta\|_{L^2}^2 \right) + 2\underline{\kappa} e^{\beta t} \|\nabla\theta\|_{L^2}^2 \leq \beta e^{\beta t} \|\theta\|_{L^2}^2 \leq \beta e^{-\beta t} \|\theta_0\|_{L^2}^2. \quad (13)$$

For any $t \geq 0$, upon integrating (13) in time over $[0, t]$ we have

$$\int_0^t e^{\beta\tau} \|\nabla\theta(\cdot, \tau)\|_{L^2}^2 d\tau \leq \alpha_0 \|\theta_0\|_{L^2}^2 \quad \forall t \geq 0, \quad (14)$$

where $\alpha_0 = \underline{\kappa}^{-1}$. This completes the proof. \square

Using this decay estimate of θ , we proceed to find the uniform estimate of $\|u\|_{H^1}$.

Lemma 3.2. *Under the assumptions of Theorem 1, it holds that*

$$\|u(\cdot, t)\|_{H^1}^2 \leq c_8 \quad \forall t \geq 0.$$

Proof. Multiplying (1)₁ by u and integrating the result equation over Ω , we have

$$\frac{d}{dt} \|u\|_{L^2}^2 = 2 \int_{\Omega} \theta \mathbf{e}_2 \cdot u d\mathbf{x}.$$

The Cauchy-Schwartz inequality and (12) imply that

$$\frac{d}{dt} \|u\|_{L^2}^2 \leq e^{-\beta t} \|u\|_{L^2}^2 + e^{\beta t} \|\theta\|_{L^2}^2$$

$$\leq e^{-\beta t} \|u\|_{L^2}^2 + e^{-\beta t} \|\theta_0\|_{L^2}^2. \quad (15)$$

Applying Gronwall's inequality to (15), we have

$$\begin{aligned} \|u(\cdot, t)\|_{L^2}^2 &\leq \exp\left(\int_0^t e^{-\beta\tau} d\tau\right) \left(\|u_0\|_{L^2}^2 + \int_0^t e^{-\beta\tau} \|\theta_0\|_{L^2}^2 d\tau\right) \\ &\leq e^{\frac{1}{\beta}} (\|u_0\|_{L^2}^2 + \beta^{-1} \|\theta_0\|_{L^2}^2) \quad \forall t \geq 0. \end{aligned} \quad (16)$$

Taking curl on both sides of (1)₁, we have

$$\omega_t + u \cdot \nabla \omega = \theta_x, \quad (17)$$

where $\omega = u_{2x} - u_{1y}$ is the 2D vorticity. Taking the L^2 inner product of (17) with ω and using Cauchy-Schwartz inequality, we have

$$\frac{d}{dt} \|\omega\|_{L^2}^2 \leq e^{-\beta t} \|\omega\|_{L^2}^2 + e^{\beta t} \|\nabla \theta\|_{L^2}^2.$$

Gronwall's inequality and Lemma 3.1 then yield

$$\begin{aligned} \|\omega(\cdot, t)\|_{L^2}^2 &\leq \exp\left(\int_0^t e^{-\beta\tau} d\tau\right) \left(\|\omega(\cdot, 0)\|_{L^2}^2 + \int_0^t e^{\beta\tau} \|\nabla \theta(\cdot, \tau)\|_{L^2}^2 d\tau\right) \\ &\leq e^{\frac{1}{\beta}} (\|\omega(\cdot, 0)\|_{L^2}^2 + \alpha_0 \|\theta_0\|_{L^2}^2) \quad \forall t \geq 0. \end{aligned} \quad (18)$$

By combining (16), (18), and Lemma 2.3 with $s = 1$, $p = 2$, we have

$$\|u(\cdot, t)\|_{H^1}^2 \leq c_8 \quad \forall t \geq 0,$$

where

$$c_8 := 2c_6^2 e^{\frac{1}{\beta}} (\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + (\beta^{-1} + \alpha_0) \|\theta_0\|_{L^2}^2).$$

This completes the proof. \square

Now we are ready to prove the exponential decay of $\|\theta\|_{H^1}$. Lemma 3.2 will play an important role.

Lemma 3.3. *Under the assumptions of Theorem 1.1, there exist a constant $\alpha_1 > 0$ independent of t such that*

$$\begin{aligned} \|\theta(\cdot, t)\|_{H^1}^2 &\leq \alpha_1 \|\theta_0\|_{H^1}^2 e^{-\beta t}, \quad \int_0^t e^{\beta\tau} \|\hat{\theta}(\cdot, \tau)\|_{H^2}^2 d\tau \leq c_{11} \|\theta_0\|_{H^1}^2, \\ \int_0^t e^{\beta\tau} \|\theta_t(\cdot, \tau)\|_{L^2}^2 d\tau &\leq c_{12} \|\theta_0\|_{H^1}^2 \quad \forall t \geq 0. \end{aligned}$$

Proof. Multiplying (8) with $\Delta \hat{\theta}$, then integrating over Ω , we have

$$\begin{aligned} \underline{\kappa} \|\Delta \hat{\theta}\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \hat{\theta}\|_{L^2}^2 &\leq \int_{\Omega} |u \cdot \nabla \hat{\theta}| |\Delta \hat{\theta}| dx \\ &\leq \frac{\underline{\kappa}}{4} \|\Delta \hat{\theta}\|_{L^2}^2 + \frac{1}{\underline{\kappa}} \|u \cdot \nabla \hat{\theta}\|_{L^2}^2. \end{aligned} \quad (19)$$

Here we have used Cauchy-Schwartz inequality. Using Lemma 2.1 and Lemma 3.2, we note that

$$\begin{aligned} \|u \cdot \nabla \hat{\theta}\|_{L^2}^2 &\leq \|u\|_{L^4}^2 \|\nabla \hat{\theta}\|_{L^4}^2 \\ &\leq c_2 c_3 \|u\|_{H^1}^2 \left(\|\nabla \hat{\theta}\|_{L^2} \|\nabla^2 \hat{\theta}\|_{L^2} + \|\nabla \hat{\theta}\|_{L^2}^2 \right) \\ &\leq \frac{\underline{\kappa}^2}{4} \|\Delta \hat{\theta}\|_{L^2}^2 + c_9 \|\nabla \theta\|_{L^2}^2, \end{aligned} \quad (20)$$

where $c_9 := \bar{\kappa}^2 c_2 c_3 c_8 (1 + \underline{\kappa}^{-2} c_2 c_3 c_5^2 c_8)$. So we update (19) as

$$\frac{d}{dt} \|\nabla \hat{\theta}\|_{L^2}^2 + \underline{\kappa} \|\Delta \hat{\theta}\|_{L^2}^2 \leq 2\underline{\kappa}^{-1} c_9 \|\nabla \theta\|_{L^2}^2. \quad (21)$$

Now, we multiply (21) by $e^{\beta t}$, we have

$$\frac{d}{dt} \left(e^{\beta t} \|\nabla \hat{\theta}\|_{L^2}^2 \right) + \underline{\kappa} e^{\beta t} \|\Delta \hat{\theta}\|_{L^2}^2 \leq (\beta \bar{\kappa} + 2\underline{\kappa}^{-1} c_9) e^{\beta t} \|\nabla \theta\|_{L^2}^2. \quad (22)$$

For any $t \geq 0$, upon integrating (22) in time over $[0, t]$ we get

$$e^{\beta t} \|\nabla \hat{\theta}(\cdot, t)\|_{L^2}^2 + \underline{\kappa} \int_0^t e^{\beta\tau} \|\Delta \hat{\theta}(\cdot, \tau)\|_{L^2}^2 d\tau \leq c_{10} \|\theta_0\|_{H^1}^2, \quad (23)$$

which gives

$$\|\nabla\theta(\cdot, t)\|_{L^2}^2 \leq \underline{\kappa}^{-2} c_{10} \|\theta_0\|_{H^1}^2 e^{-\beta t}, \quad \int_0^t e^{\beta\tau} \|\nabla^2 \hat{\theta}(\cdot, \tau)\|_{L^2}^2 d\tau \leq \underline{\kappa}^{-1} c_5^2 c_{10} \|\theta_0\|_{H^1}^2 \quad \forall t \geq 0, \quad (24)$$

here $c_{10} := \bar{\kappa}^2 + (\beta\bar{\kappa} + 2c_9\underline{\kappa}^{-1})\alpha_0$.

Therefore, by using Lemma 3.1, we have

$$\|\theta(\cdot, t)\|_{H^1}^2 \leq \alpha_1 \|\theta_0\|_{H^1}^2 e^{-\beta t}, \quad \int_0^t e^{\beta\tau} \|\hat{\theta}(\cdot, \tau)\|_{H^2}^2 d\tau \leq c_{11} \|\theta_0\|_{H^1}^2, \quad (25)$$

where $\alpha_1 := 1 + \underline{\kappa}^{-2} c_{10}$, $c_{11} := \bar{\kappa}^2(1 + c_0)\alpha_0 + \underline{\kappa}^{-1} c_5^2 c_{10}$.

On the other hand, taking (8) by the L^2 norm, and using (20), (23), we can update it as

$$\|\hat{\theta}_t\|_{L^2}^2 \leq \left(\bar{\kappa}^2 + \frac{\underline{\kappa}^2}{4} \right) \|\Delta\hat{\theta}\|_{L^2}^2 + c_9 \|\nabla\theta\|_{L^2}^2.$$

Then, we multiply the above inequality by $e^{\beta t}$, upon integrating the result inequality in time over $[0, t]$, we obtain

$$\int_0^t e^{\beta\tau} \|\theta_t(\cdot, \tau)\|_{L^2}^2 d\tau \leq c_{12} \|\theta_0\|_{H^1}^2, \quad (26)$$

where $c_{12} := \underline{\kappa}^{-2} (c_9\alpha_0 + \underline{\kappa}^{-1} c_{10} (\bar{\kappa}^2 + 4^{-1}\underline{\kappa}^2))$. This completes the proof. \square

With the help of Lemma 3.3, we will establish a higher-order uniform estimate of u in the next lemma.

Lemma 3.4. *Under the assumptions of Theorem 1.1, for any $p \in [2, \infty)$, it holds that*

$$\|u(\cdot, t)\|_{W^{1,p}} \leq c_{13}, \quad \|u_t(\cdot, t)\|_{L^p} \leq c_{14} \quad \forall t \geq 0.$$

Proof. For the first estimate, by virtue of Lemma 2.3 and Lemma 3.2, we will only need to prove the uniform estimate on $\|\omega\|_{L^p}$. For any fixed $p \in [2, \infty)$, multiplying (17) with

$|\omega|^{p-2}\omega$, integrating over Ω , one has

$$\frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p}^p = \int_{\Omega} \theta_x |\omega|^{p-2} \omega dx \leq \|\nabla \theta\|_{L^p} \|\omega\|_{L^p}^{p-1}. \quad (27)$$

This implies that

$$\frac{d}{dt} \|\omega\|_{L^p} \leq \|\nabla \theta\|_{L^p} \leq \underline{\kappa}^{-1} \|\nabla \hat{\theta}\|_{L^p} \leq \underline{\kappa}^{-1} c_2 \|\hat{\theta}\|_{H^2}. \quad (28)$$

Upon integrating in time and using Hölder inequality we have

$$\begin{aligned} \|\omega(\cdot, t)\|_{L^p} &\leq \|\omega(\cdot, 0)\|_{L^p} + \underline{\kappa}^{-1} c_2 \int_0^t \|\hat{\theta}(\cdot, \tau)\|_{H^2} d\tau \\ &\leq \|\omega(\cdot, 0)\|_{L^p} + \underline{\kappa}^{-1} c_2 \left(\int_0^t e^{-\beta\tau} d\tau \right)^{\frac{1}{2}} \left(\int_0^t e^{\beta\tau} \|\hat{\theta}(\cdot, \tau)\|_{H^2}^2 d\tau \right)^{\frac{1}{2}} \\ &\leq \|\omega(\cdot, 0)\|_{L^p} + \underline{\kappa}^{-1} \beta^{-\frac{1}{2}} c_2 c_{11}^{\frac{1}{2}} \|\theta_0\|_{H^1}. \end{aligned} \quad (29)$$

Therefore, Lemma 2.3, Lemma 3.2 and Lemma 2.1 imply that for any $p \in [2, \infty)$,

$$\begin{aligned} \|u(\cdot, t)\|_{W^{1,p}} &\leq c_6 (\|\omega(\cdot, t)\|_{L^p} + \|u(\cdot, t)\|_{L^p}) \\ &\leq c_6 \left(\|\omega(\cdot, t)\|_{L^p} + c_2^{\frac{1}{2}} \|u(\cdot, t)\|_{H^1} \right) \\ &\leq c_{13}, \end{aligned} \quad (30)$$

where $c_{13} := c_6 \left(\|\omega_0\|_{L^p} + \underline{\kappa}^{-1} \beta^{-\frac{1}{2}} c_2 c_{11}^{\frac{1}{2}} \|\theta_0\|_{H^1} + c_2^{\frac{1}{2}} c_8^{\frac{1}{2}} \right)$.

Then, we turn to the second estimate. Multiplying (1)₁ by $|u_t|^{p-2} u_t$ and integrating on Ω we get

$$\begin{aligned} \|u_t\|_{L^p}^p &= \int_{\Omega} \theta e_2 \cdot |u_t|^{p-2} u_t dx - \int_{\Omega} (u \cdot \nabla u) \cdot |u_t|^{p-2} u_t dx \\ &\leq \frac{1}{2} \|u_t\|_{L^p}^p + 4^{p-1} (\|\theta\|_{L^p}^p + \|u \cdot \nabla u\|_{L^p}^p) \\ &\leq \frac{1}{2} \|u_t\|_{L^p}^p + 4^{p-1} \left(c_2^{\frac{p}{2}} \|\theta\|_{H^1}^p + c_1^{\frac{p}{2}} \|u\|_{W^{1,3}}^p \|\nabla u\|_{L^p}^p \right). \end{aligned} \quad (31)$$

Using Lemma 3.3, (30) and (31) we have

$$\|u_t(\cdot, t)\|_{L^p} \leq c_{14} \quad \forall t \geq 0, \quad (32)$$

where $c_{14} := 2^{\frac{2p-1}{p}} \left(\alpha_1^{\frac{p}{2}} c_2^{\frac{p}{2}} \|\theta_0\|_{H^1}^p + c_1^{\frac{p}{2}} c_{13}^{2p} \right)$. This completes the proof. \square

We are now ready to improve the decay estimate of θ to higher order by virtue of Lemma 3.4.

Lemma 3.5. *Under the assumptions of Theorem 1.1, there exist a constant $\alpha_2 > 0$ independent of t such that*

$$\begin{aligned} \|(\theta, \hat{\theta})(\cdot, t)\|_{H^2}^2 &\leq \alpha_2 e^{-\beta t}, \quad \|\theta_t(\cdot, t)\|_{L^2}^2 \leq c_{18} e^{-\beta t}, \\ \int_0^t e^{\beta\tau} \|(\nabla\theta_t, \nabla\hat{\theta}_t)(\cdot, \tau)\|_{L^2}^2 d\tau &\leq c_{22} \quad \forall t \geq 0. \end{aligned}$$

Proof. Taking the temporal derivative of (1)₂ we have

$$\theta_{tt} + u_t \cdot \nabla\theta + u \cdot \nabla\theta_t = \nabla \cdot (\kappa(\theta)\nabla\theta_t) + \nabla \cdot (\kappa'(\theta)\theta_t\nabla\theta). \quad (33)$$

Multiplying (33) by θ_t and integrating on Ω we have

$$\frac{1}{2} \frac{d}{dt} \|\theta_t\|_{L^2}^2 + \underline{\kappa} \|\nabla\theta_t\|_{L^2}^2 \leq - \int_{\Omega} (u_t \cdot \nabla\theta) \theta_t d\mathbf{x} - \int_{\Omega} \kappa'(\theta)\theta_t (\nabla\theta \cdot \nabla\theta_t) d\mathbf{x}. \quad (34)$$

The first term on the RHS of (34) can be estimated as

$$\begin{aligned} \left| - \int_{\Omega} (u_t \cdot \nabla\theta) \theta_t d\mathbf{x} \right| &= \left| \int_{\Omega} \theta u_t \cdot \nabla\theta_t d\mathbf{x} \right| \\ &\leq \frac{\underline{\kappa}}{6} \|\nabla\theta_t\|_{L^2}^2 + \frac{3}{2\underline{\kappa}} \|\theta u_t\|_{L^2}^2 \\ &\leq \frac{\underline{\kappa}}{6} \|\nabla\theta_t\|_{L^2}^2 + \frac{3}{2\underline{\kappa}^2} \|\hat{\theta}\|_{L^\infty}^2 \|u_t\|_{L^2}^2 \\ &\leq \frac{\underline{\kappa}}{6} \|\nabla\theta_t\|_{L^2}^2 + \frac{3}{2\underline{\kappa}^2} c_1 c_2 c_{14}^2 \|\hat{\theta}\|_{H^2}^2. \end{aligned} \quad (35)$$

The second term on the RHS of (34) can be estimated as

$$\left| - \int_{\Omega} \kappa'(\theta)\theta_t \nabla\theta \cdot \nabla\theta_t d\mathbf{x} \right|$$

$$\begin{aligned}
&\leq \frac{\underline{\kappa}}{6} \|\nabla\theta_t\|_{L^2}^2 + \frac{3}{2}\underline{\kappa}^{-1}(\max\{|\kappa'(\theta)|\})^2 \|\theta_t \nabla\theta\|_{L^2}^2 \\
&\leq \frac{\underline{\kappa}}{6} \|\nabla\theta_t\|_{L^2}^2 + \frac{3}{2}\underline{\kappa}^{-1}(\max\{|\kappa'(\theta)|\})^2 c_3 c_4 \|\theta_t\|_{L^2} \|\nabla\theta_t\|_{L^2} \left(\|\nabla\hat{\theta}\|_{L^2} \|\nabla^2\hat{\theta}\|_{L^2} + \|\nabla\hat{\theta}\|_{L^2}^2 \right) \\
&\leq \frac{\underline{\kappa}}{3} \|\nabla\theta_t\|_{L^2}^2 + c_{15} \left(\|\hat{\theta}\|_{H^2}^2 + \bar{\kappa}^2 \|\nabla\theta\|_{L^2}^2 \right) \|\theta_t\|_{L^2}^2, \tag{36}
\end{aligned}$$

where $c_{15} := \frac{27}{4}\bar{\kappa}^2\underline{\kappa}^{-3}(\max\{|\kappa'(\theta)|\})^4\alpha_1c_3^2c_4^2\|\theta_0\|_{H^1}^2$.

Plugging (35)-(36) into (34), we have

$$\frac{d}{dt} \|\theta_t\|_{L^2}^2 + \underline{\kappa} \|\nabla\theta_t\|_{L^2}^2 \leq 2c_{15} \left(\|\hat{\theta}\|_{H^2}^2 + \bar{\kappa}^2 \|\nabla\theta\|_{L^2}^2 \right) \|\theta_t\|_{L^2}^2 + c_{16} \|\hat{\theta}\|_{H^2}^2, \tag{37}$$

where $c_{16} := 3\underline{\kappa}^{-2}c_1c_2c_{14}^2$. Using Gronwall's inequality to the above inequality, we have

$$\begin{aligned}
\|\theta_t(\cdot, t)\|_{L^2}^2 &\leq \exp\left(2c_{15} \int_0^t \left(\|\hat{\theta}\|_{H^2}^2 + \bar{\kappa}^2 \|\nabla\theta\|_{L^2}^2\right) d\tau\right) \left(\|\theta_t(\cdot, 0)\|_{L^2}^2 + c_{16} \int_0^t \|\hat{\theta}(\cdot, \tau)\|_{H^2}^2 d\tau\right) \\
&\leq c_{17} \quad \forall t \geq 0, \tag{38}
\end{aligned}$$

where $c_{17} := \exp\left(2c_{15} \left(c_{11}\|\theta_0\|_{H^1}^2 + \bar{\kappa}^2\alpha_0\|\theta_0\|_{L^2}^2\right)\right) \left(\|\theta_t(\cdot, 0)\|_{L^2}^2 + c_{11}c_{16}\|\theta_0\|_{H^1}^2\right)$.

Next, we update (37) as (by using (38))

$$\frac{d}{dt} \|\theta_t\|_{L^2}^2 + \underline{\kappa} \|\nabla\theta_t\|_{L^2}^2 \leq (2c_{15}c_{17} + c_{16}) \|\hat{\theta}\|_{H^2}^2 + 2\bar{\kappa}^2c_{15}c_{17} \|\nabla\theta\|_{L^2}^2. \tag{39}$$

Multiplying (39) by $e^{\beta t}$, and integrating the result inequality in time over $[0, t]$ we have

$$e^{\beta t} \|\theta_t(\cdot, t)\|_{L^2}^2 + \underline{\kappa} \int_0^t e^{\beta\tau} \|\nabla\theta_t(\cdot, \tau)\|_{L^2}^2 d\tau \leq c_{18} \quad \forall t \geq 0, \tag{40}$$

where $c_{18} := ((2c_{15}c_{17} + c_{16})c_{11} + \beta c_{12} + 2\bar{\kappa}^2\alpha_1c_{15}c_{17}) \|\theta_0\|_{H^1}^2 + \|\theta_t(\cdot, 0)\|_{L^2}^2$.

This implies that

$$\|\theta_t(\cdot, t)\|_{L^2}^2 \leq c_{18}e^{-\beta t}, \quad \int_0^t e^{\beta\tau} \|\nabla\theta_t(\cdot, \tau)\|_{L^2}^2 d\tau \leq \underline{\kappa}^{-1}c_{18} \quad \forall t \geq 0. \tag{41}$$

By virtue of Lemma 2.2 and (8) we have

$$\begin{aligned}
\|\hat{\theta}(\cdot, t)\|_{H^2}^2 &\leq 2c_6^2 \left(\|\theta_t(\cdot, t)\|_{L^2}^2 + \|u \cdot \nabla\theta\|_{L^2}^2 \right) \\
&\leq 2c_6^2 \left(c_{18}e^{-\beta t} + \alpha_1c_1c_{13}^2e^{-\beta t} \right)
\end{aligned}$$

$$\leq c_{19}e^{-\beta t} \quad \forall t \geq 0, \quad (42)$$

where $c_{19} := 2c_6^2 (c_{18} + \alpha_1 c_1 c_{13}^2)$.

Using the previous lemmas we have

$$\begin{aligned} \|\nabla^2 \theta\|_{L^2} &= \|\kappa^{-1}(\theta)\nabla^2 \hat{\theta} - \kappa^{-3}(\theta)\kappa'(\theta)\nabla \hat{\theta} \otimes \nabla \hat{\theta}\|_{L^2} \\ &\leq \underline{\kappa}^{-1} \|\hat{\theta}\|_{H^2} + \underline{\kappa}^{-3} \max\{|\kappa'(\theta)|\} c_2 \|\hat{\theta}\|_{H^2}^2 \\ &\leq c_{20} e^{-\frac{\beta}{2}t}, \end{aligned} \quad (43)$$

where

$$c_{20} := \underline{\kappa}^{-1} c_{19}^{\frac{1}{2}} \left(1 + \underline{\kappa}^{-2} \max\{|\kappa'(\theta)|\} c_2 c_{19}^{\frac{1}{2}} \right).$$

Therefore, using Lemma 3.3, (42) and (43) we have

$$\|(\theta, \hat{\theta})(\cdot, t)\|_{H^2}^2 \leq \alpha_2 e^{-\beta t} \quad \forall t \geq 0,$$

where $\alpha_2 := \max\{\alpha_1 \|\theta_0\|_{H^1}^2 + c_{20}^2, c_{19}\}$. In virtue of the previous lemmas, we get

$$\begin{aligned} \|\nabla \hat{\theta}_t\|_{L^2}^2 &= \|\kappa'(\theta)\theta_t \nabla \theta + \kappa(\theta)\nabla \theta_t\|_{L^2}^2 \\ &\leq 2 \left(\max\{|\kappa'(\theta)|\} \right)^2 \|\nabla \theta\|_{L^4}^2 \|\nabla \theta_t\|_{L^2}^2 + 2\bar{\kappa}^2 \|\nabla \theta_t\|_{L^2}^2 \\ &\leq c_{21} \|\nabla \theta_t\|_{L^2}^2, \end{aligned} \quad (44)$$

where

$$c_{21} := 2 \left(\bar{\kappa}^2 + \alpha_2 c_2 \left(\max\{|\kappa'(\theta)|\} \right)^2 \right).$$

This, together with (41), implies that

$$\int_0^t e^{\beta\tau} \|(\nabla \theta_t, \nabla \hat{\theta}_t)(\cdot, \tau)\|_{L^2}^2 d\tau \leq c_{22} \quad \forall t \geq 0,$$

where $c_{22} = (1 + c_{21})c_{18}\bar{\kappa}^{-1}$. This completes the proof. \square

We will prove the decay estimate of $\|\theta_t\|_{H^1}^2$ in the next lemma, which will then imply the decay estimate of $\|\theta\|_{H^3}^2$.

Lemma 3.6. *Under the assumptions of Theorem 1.1, there exist a constant $\alpha_3 > 0$ independent of t such that*

$$\begin{aligned} \|(\theta_t, \hat{\theta}_t)(\cdot, t)\|_{H^1}^2 &\leq \alpha_3 e^{-\beta t}, \\ \int_0^t e^{\beta\tau} \|(\theta_{tt}, \hat{\theta}_{tt})(\cdot, \tau)\|_{L^2}^2 d\tau &\leq c_{29} \quad \forall t \geq 0. \end{aligned}$$

Proof. Taking the temporal derivative of (8) we have

$$\Delta \hat{\theta}_t = -\kappa'(\theta)\kappa^{-1}(\theta)\theta_t \Delta \hat{\theta} + \kappa^{-1}(\theta) \left(\hat{\theta}_{tt} + u_t \cdot \nabla \hat{\theta} + u \cdot \nabla \hat{\theta}_t \right). \quad (45)$$

Multiplying (45) by $\hat{\theta}_{tt}$ and integrating over Ω we get

$$\begin{aligned} &\frac{1}{\bar{\kappa}} \|\hat{\theta}_{tt}\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \hat{\theta}_t\|_{L^2}^2 \\ &\leq \int_{\Omega} \kappa^{-1}(\theta) |\hat{\theta}_{tt}| \left(|u_t \cdot \nabla \hat{\theta}| + |u \cdot \nabla \hat{\theta}_t| \right) dx + \int_{\Omega} |\kappa'(\theta)| \kappa^{-1}(\theta) |\theta_t| |\Delta \hat{\theta}| |\hat{\theta}_{tt}| dx \\ &:= \sum_{j=1}^2 I_j \end{aligned} \quad (46)$$

We can estimate each term on the RHS of (46) as follows

$$\begin{aligned} |I_1| &\leq \frac{1}{4\bar{\kappa}} \|\hat{\theta}_{tt}\|_{L^2}^2 + 2\bar{\kappa}\underline{\kappa}^{-2} \left(\|u_t \cdot \nabla \hat{\theta}\|_{L^2}^2 + \|u \cdot \nabla \hat{\theta}_t\|_{L^2}^2 \right) \\ &\leq \frac{1}{4\bar{\kappa}} \|\hat{\theta}_{tt}\|_{L^2}^2 + 2\bar{\kappa}\underline{\kappa}^{-2} \left(\|u_t\|_{L^4}^2 \|\nabla \hat{\theta}\|_{L^4}^2 + \|u\|_{L^\infty}^2 \|\nabla \hat{\theta}_t\|_{L^2}^2 \right) \\ &\leq \frac{1}{4\bar{\kappa}} \|\hat{\theta}_{tt}\|_{L^2}^2 + 2\bar{\kappa}\underline{\kappa}^{-2} c_2 c_{14}^2 \|\hat{\theta}\|_{H^2}^2 + 2\bar{\kappa}\underline{\kappa}^{-2} c_1 c_{13}^2 \|\nabla \hat{\theta}_t\|_{L^2}^2, \end{aligned} \quad (47)$$

$$\begin{aligned} |I_2| &\leq \frac{1}{4\bar{\kappa}} \|\hat{\theta}_{tt}\|_{L^2}^2 + 2\bar{\kappa}\underline{\kappa}^{-2} \left(\max\{|\kappa'(\theta)|\} \right)^2 \left(\|\theta_t\|_{L^4}^4 + \|\theta_t(u \cdot \nabla \theta)\|_{L^2}^2 \right) \\ &\leq \frac{1}{4\bar{\kappa}} \|\hat{\theta}_{tt}\|_{L^2}^2 + 2\bar{\kappa}\underline{\kappa}^{-2} \left(\max\{|\kappa'(\theta)|\} \right)^2 \left(2\|\theta_t\|_{L^4}^4 + \|u \cdot \nabla \theta\|_{L^4}^4 \right) \\ &\leq \frac{1}{4\bar{\kappa}} \|\hat{\theta}_{tt}\|_{L^2}^2 + 2\bar{\kappa}\underline{\kappa}^{-2} \left(\max\{|\kappa'(\theta)|\} \right)^2 \left(2c_4^2 \|\theta_t\|_{L^2}^2 \|\nabla \theta_t\|_{L^2}^2 + \|u\|_{L^8}^4 \|\nabla \theta\|_{L^8}^4 \right) \end{aligned}$$

$$\leq \frac{1}{4\bar{\kappa}} \|\hat{\theta}_{tt}\|_{L^2}^2 + 2\bar{\kappa}\underline{\kappa}^{-2} \left(\max\{|\kappa'(\theta)|\} \right)^2 \left(2c_4^2 c_{18} \|\nabla\theta_t\|_{L^2}^2 + c_4^2 c_{13}^2 \|\theta\|_{H^2}^4 \right), \quad (48)$$

where we have used the equation (1.1)₂.

Plugging (47)-(48) into (46), we have

$$\|\hat{\theta}_{tt}\|_{L^2}^2 + \bar{\kappa} \frac{d}{dt} \|\nabla\hat{\theta}_t\|_{L^2}^2 \leq c_{23} \|(\nabla\theta_t, \nabla\hat{\theta}_t)\|_{L^2}^2 + c_{24} \|\hat{\theta}\|_{H^2}^2 + c_{25} \|\theta\|_{H^2}^4, \quad (49)$$

where

$$\begin{aligned} c_{23} &:= 4\bar{\kappa}^2 \underline{\kappa}^{-2} \left(c_1 c_{13}^2 + 2c_4^2 c_{18} \max\{(\kappa'(\theta))^2\} \right), \\ c_{24} &:= 4\bar{\kappa}^2 \underline{\kappa}^{-2} c_2 c_{14}^2, \\ c_{25} &:= 4\bar{\kappa}^2 \underline{\kappa}^{-2} c_4^2 c_{13}^2 \max\{(\kappa'(\theta))^2\}. \end{aligned}$$

Then, we multiply (49) by $e^{\beta t}$, and integrate the result inequality in time over $[0, t]$, we have

$$\bar{\kappa} e^{\beta t} \|\nabla\hat{\theta}_t(\cdot, t)\|_{L^2}^2 + \int_0^t e^{\beta\tau} \|\hat{\theta}_{tt}(\cdot, \tau)\|_{L^2}^2 d\tau \leq c_{26} \quad \forall t \geq 0, \quad (50)$$

where $c_{26} := c_{22} c_{23} \|\theta_0\|_{H^1}^2 + \alpha_2 c_{24} + \alpha_2 c_{25} + c_{21} \|\nabla\theta_t(\cdot, 0)\|_{L^2}^2$.

Similar to (44), one has

$$\|\nabla\theta_t\|_{L^2}^2 \leq c_{27} \|\nabla\hat{\theta}_t\|_{L^2}^2, \quad (51)$$

where $c_{27} := 2\underline{\kappa}^{-2} \left(1 + \underline{\kappa}^{-2} \alpha_2 c_2^2 (c_0 + 1) \left(\max\{|\underline{\kappa}'(\theta)|\} \right)^2 \right)$.

Employing (50), (51) and Lemma 3.5, one can easily obtain

$$\|(\theta_t, \hat{\theta}_t)(\cdot, t)\|_{H^1}^2 \leq \alpha_3 e^{-\beta t} \quad \forall t \geq 0, \quad (52)$$

where $\alpha_3 := \max\{c_{18} + \bar{\kappa}^{-1} c_{27} c_{26}, c_{18} \bar{\kappa}^2 + c_{26} \bar{\kappa}^{-1}\}$.

We note that

$$\begin{aligned}\|\theta_{tt}\|_{L^2}^2 &= \|\kappa^{-1}(\theta)\hat{\theta}_{tt} - \kappa^{-1}(\theta)\kappa'(\theta)\theta_t^2\|_{L^2}^2 \\ &\leq 2\underline{\kappa}^{-2}\|\hat{\theta}_{tt}\|_{L^2}^2 + 2c_2^2\underline{\kappa}^{-2}\max\{(\kappa'(\theta))^2\}\|\theta_t\|_{H^1}^4.\end{aligned}$$

Combining (50), (52) and the above inequality, one can easily obtain

$$\int_0^t e^{\beta\tau} \|(\theta_{tt}, \hat{\theta}_{tt})(\cdot, \tau)\|_{L^2}^2 d\tau \leq c_{28} \quad \forall t \geq 0, \quad (53)$$

where $c_{28} := \max\{c_{26}, 2\underline{\kappa}^{-2}(c_{26} + c_2^2 \max\{(\kappa'(\theta))^2\}\beta^{-1})\}$.

This completes the proof. \square

We are now ready to complete the decay estimate of $\|\theta\|_{H^3}^2$.

Lemma 3.7. *Under the assumptions of Theorem 1.1, there exist a constant $\alpha > 0$ independent of t such that*

$$\|(\theta, \hat{\theta})(\cdot, t)\|_{H^3}^2 \leq \alpha e^{-\beta t} \quad \forall t \geq 0.$$

Proof. First, we can write the equation of (1)₂ as

$$\Delta \hat{\theta} = \theta_t + u \cdot \nabla \theta. \quad (54)$$

Using Lemma 2.2 and (54) we have

$$\|\hat{\theta}\|_{H^3}^2 \leq 2c_5^2 (\|\theta_t\|_{H^1}^2 + \|u \cdot \nabla \theta\|_{H^1}^2). \quad (55)$$

We can estimate the second term on the RHS of (55) as

$$\begin{aligned}&\|u \cdot \nabla \theta\|_{H^1}^2 \\ &\leq \|u \cdot \nabla \theta\|_{L^2}^2 + 2\|\nabla u \cdot \nabla \theta\|_{L^2}^2 + 2\|u \cdot \nabla^2 \theta\|_{L^2}^2 \\ &\leq (c_2\|u\|_{L^4}^2 + 2c_2\|\nabla u\|_{L^4}^2 + 2c_1\|u\|_{W^{1,p}}^2) \|\theta\|_{H^2}^2 \\ &\leq \alpha_2 c_{13}^2 (3c_2 + 2c_1) e^{-\beta t}.\end{aligned} \quad (56)$$

Employing estimates (52) and (56), from (55) we obtain

$$\|\hat{\theta}(\cdot, t)\|_{H^3}^2 \leq c_{29} e^{-\beta t} \quad \forall t > 0, \quad (57)$$

where $c_{29} := 2c_5^2 (\alpha_3 + \alpha_2 c_{13}^2 (2c_1 + 3c_2))$.

Using Lemma 2.4 and the previous lemmas, we have

$$\begin{aligned} \|\nabla^3 \theta\|_{L^2}^2 &= \|\nabla^2(\kappa^{-1}(\theta)\nabla\hat{\theta})\|_{L^2}^2 \\ &\leq 2c_7^2 \left(\|\kappa^{-1}(\theta)\|_{L^\infty}^2 \|\nabla^3 \hat{\theta}\|_{L^2}^2 + \|\nabla\hat{\theta}\|_{L^\infty}^2 \|\nabla^2(\kappa^{-1}(\theta))\|_{L^2}^2 \right)^2 \\ &\leq c_{30} \|\hat{\theta}\|_{H^3}^2, \end{aligned} \quad (58)$$

where

$$c_{30} := 2c_7^2 \left(\underline{\kappa}^{-2} + c_1 c_2 \alpha_2 (\alpha_2 c_2 \max\{((\kappa^{-2}(\theta)\kappa'(\theta))')^2\} + \max\{(\kappa^{-2}(\theta)\kappa'(\theta))^2\}) \right).$$

Combining Lemma 3.5, (57) and (58), we have

$$\|(\theta, \hat{\theta})(\cdot, t)\|_{H^3}^2 \leq \alpha e^{-\beta t} \quad \forall t \geq 0, \quad (59)$$

where $\alpha := \max\{c_{29}, \alpha_2 + c_{29}c_{30}\}$. This completes the proof. \square

As an immediate consequence of Lemma 3.7, one has

Lemma 3.8. *Under the assumptions of Theorem 1.1, it holds that*

$$\|\omega(\cdot, t)\|_{L^\infty} \leq c_{31} \quad \forall t \geq 0.$$

Proof. We note from (28) that for any $p \geq 2$, it holds that

$$\frac{d}{dt} \|\omega\|_{L^p} \leq \|\nabla\theta\|_{L^p} \leq \|\nabla\theta\|_{L^\infty} |\Omega|^{\frac{1}{p}} \leq \max\{1, |\Omega|\} c_1^{\frac{1}{2}} c_2^{\frac{1}{2}} \|\theta\|_{H^3}.$$

Using Lemma 3.7 and integrating the above inequality in time, we have

$$\|\omega(\cdot, t)\|_{L^p} \leq c_{31},$$

where $c_{31} := \|\omega_0\|_{L^p} + \max\{1, |\Omega|\} c_1^{\frac{1}{2}} c_2^{\frac{1}{2}} \alpha \beta^{-1}$.

Letting $p \rightarrow \infty$ we get

$$\|\omega(\cdot, t)\|_{L^\infty} \leq c_{31} \quad \forall t \geq 0.$$

This completes the proof. \square

Now we turn to the regularity of the velocity field.

Lemma 3.9. *Under the assumptions of Theorem 1.1, there exists a constant $M > 0$ depending on T and other constants indicated in previous lemmas such that*

$$\|u\|_{C([0,T];H^3(\Omega))}^2 \leq M(T) < \infty \quad \text{for all } 0 < T < \infty.$$

Proof. First, we note that, by Lemma 3.8 and Sobolev embedding

$$\|\theta\|_{C([0,T];C^{1+\gamma}(\bar{\Omega}))}^2 \leq C$$

for some $\gamma \in (0, 1)$. Therefore, (1)₁ and Lemma 2.5 imply that for any fixed $T > 0$,

$$\|u\|_{C([0,T];C^1(\bar{\Omega}))}^2 \leq C. \quad (60)$$

By virtue of Lemma 2.3 and Lemma 3.4, it suffices to show the estimate of $\|\omega\|_{H^2}^2$ in order to prove this lemma. We consider the vorticity equation (17). Taking the L^2 inner product of $\nabla(17)$ with $\nabla\omega$ we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla\omega\|_{L^2}^2 &= \int_{\Omega} \nabla\theta_x \cdot \nabla\omega \, d\mathbf{x} - \int_{\Omega} (\nabla u \cdot \nabla\omega) \cdot \nabla\omega \, d\mathbf{x} \\ &\leq \|\theta\|_{H^2}^2 + (\|\nabla u\|_{L^\infty} + 1) \|\nabla\omega\|_{L^2}^2 \\ &\leq C \|\nabla\omega\|_{L^2}^2 + C, \end{aligned} \quad (61)$$

which is due to Lemma 3.5 and (60). Applying Gronwall's inequality to (61) we have

$$\|\nabla\omega\|_{L^2}^2 \leq C(T). \quad (62)$$

This implies that

$$\|u\|_{H^2}^2 \leq C(T).$$

Similarly, taking the L^2 inner product of $\nabla^2(17)$ with $\nabla^2\omega$ we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla^2\omega\|_{L^2}^2 &= \int_{\Omega} (\nabla^2\theta_x \cdot \nabla^2\omega - (\nabla^2u \cdot \nabla\omega) \cdot \nabla^2\omega - 2(\nabla^2\omega \cdot \nabla u) \cdot \nabla^2\omega) dx \\ &\leq \|\theta\|_{H^3}^2 + C(\|\nabla u\|_{L^\infty} + 1) \|\nabla^2\omega\|_{L^2}^2 + \|\nabla^2u\nabla\omega\|_{L^2}^2. \end{aligned} \quad (63)$$

We can estimate the last term on the RHS of (63) by using Lemma 2.3 as follows

$$\begin{aligned} \|\nabla^2u\nabla\omega\|_{L^2}^2 &\leq \|\nabla^2u\|_{L^4}^2 \|\nabla\omega\|_{L^4}^2 \\ &\leq C\|\nabla^2u\|_{L^2} \|\nabla^2u\|_{H^1} \|\nabla\omega\|_{L^2} \|\nabla\omega\|_{H^1} \\ &\leq C\|u\|_{H^2} \|u\|_{H^3} \|\omega\|_{H^1} \|\omega\|_{H^2} \\ &\leq C(\|u\|_{H^2}^2 \|u\|_{H^3}^2 + \|\omega\|_{H^1}^2 \|\omega\|_{H^2}^2) \\ &\leq C(\|u\|_{H^3}^2 + \|\omega\|_{H^2}^2) \\ &\leq C\|\nabla^2\omega\|_{L^2}^2 + C(T). \end{aligned} \quad (64)$$

So we conclude from (63) and (64) that

$$\frac{1}{2} \frac{d}{dt} \|\nabla^2\omega\|_{L^2}^2 \leq C\|\nabla^2\omega\|_{L^2}^2 + C(T).$$

Applying Gronwall's inequality to the above inequality we have

$$\|\nabla^2\omega\|_{L^2}^2 \leq C(T). \quad (65)$$

Combining (18), (62) and (65) we have

$$\|\omega\|_{H^2}^2 \leq C(T). \quad (66)$$

Using Lemma 2.3, Lemma 3.2 and (66), we complete the proof of this lemma. \square

We are now ready to complete the proof of the regularity stated in Theorem 3.1, it only remains to estimate $\|\theta\|_{L^2(0,t;H^4(\Omega))}^2$. The proof is straightforward by using the results obtained in previous lemmas.

Lemma 3.10. *Under the assumptions of Theorem 1.1, it holds that*

$$\int_0^t e^{\beta\tau} \|(\theta, \hat{\theta})(\cdot, \tau)\|_{H^4}^2 d\tau \leq C \quad \forall t \geq 0.$$

Proof. Applying Lemma 2.2 to (54) we have

$$\|\hat{\theta}\|_{H^4}^2 \leq C (\|\theta_t\|_{H^2}^2 + \|u \cdot \nabla \theta\|_{H^2}^2). \quad (67)$$

Now, we shall treat each term on the RHS of (67).

First, taking the temporal derivative of (54), we have

$$\Delta \hat{\theta}_t = \theta_{tt} + u_t \cdot \nabla \theta + u \cdot \nabla \theta_t. \quad (68)$$

Applying Lemma 2.2 to (68) we have

$$\begin{aligned} \|\hat{\theta}_t\|_{H^2}^2 &\leq C (\|\theta_{tt}\|_{L^2}^2 + \|u_t \cdot \nabla \theta\|_{L^2}^2 + \|u \cdot \nabla \theta_t\|_{L^2}^2) \\ &\leq C (\|\theta_{tt}\|_{L^2}^2 + \|u_t\|_{L^4}^2 \|\theta\|_{H^2}^2 + \|u\|_{W^{1,p}}^2 \|\theta_t\|_{H^1}^2) \\ &\leq C \|\theta_{tt}\|_{L^2}^2 + C (\|\theta\|_{H^2}^2 + \|\theta_t\|_{H^1}^2). \end{aligned} \quad (69)$$

Multiplying (69) by $e^{\beta t}$, and integrating the result inequality in time over $[0, t]$, we have (by using Lemma 3.6)

$$\int_0^t e^{\beta\tau} \|\hat{\theta}_t(\cdot, \tau)\|_{H^2}^2 d\tau \leq C \quad \forall t \geq 0. \quad (70)$$

Similar to (58), we have

$$\|\theta_t\|_{H^2}^2 \leq C \|\hat{\theta}_t\|_{H^2}^2, \quad \|u \cdot \nabla \theta\|_{H^2}^2 \leq C \|\theta\|_{H^3}^2, \quad \|\theta\|_{H^4}^2 \leq C \|\hat{\theta}\|_{H^4}^2. \quad (71)$$

This implies that

$$\int_0^t e^{\beta\tau} \|(\theta_t, \hat{\theta}_t)(\cdot, \tau)\|_{H^2}^2 d\tau \leq C \quad \forall t \geq 0. \quad (72)$$

Combining (67), (71) and (72), we obtain

$$\int_0^t e^{\beta\tau} \|(\theta, \hat{\theta})(\cdot, \tau)\|_{H^4}^2 d\tau \leq C \quad \forall t \geq 0.$$

This completes the proof of Lemma 3.10. \square

4. UNIQUENESS

With the global regularity established in the previous section, we are able to prove the uniqueness of the solution in this section.

Theorem 4.1. *Under the assumptions of Theorem 1.1, the solution of (1)-(2) is unique.*

Proof. For any fixed $T > 0$, suppose there are two solutions (θ_1, u_1, P_1) and (θ_2, u_2, P_2) to (1)-(2). Setting $\bar{\theta} = \theta_1 - \theta_2$, $\bar{u} = u_1 - u_2$ and $\bar{P} = P_1 - P_2$, then $(\bar{\theta}, \bar{u}, \bar{P})$ satisfy

$$\left\{ \begin{array}{l} \bar{u}_t + u_1 \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_2 + \nabla \bar{P} = \bar{\theta} \mathbf{e}_2, \\ \bar{\theta}_t + u_1 \cdot \nabla \bar{\theta} + \bar{u} \cdot \nabla \theta_2 = \nabla \cdot (\kappa(\theta_1) \nabla \bar{\theta}) + \nabla \cdot ((\kappa(\theta_1) - \kappa(\theta_2)) \nabla \theta_2), \\ \nabla \cdot \bar{u} = 0, \\ \bar{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \bar{\theta}|_{\partial\Omega} = 0, \\ \bar{u}(x, 0) = 0, \quad \bar{\theta}(x, 0) = 0. \end{array} \right. \quad (73)$$

Taking the L^2 inner product of (73)₁ with \bar{u} and (73)₂ with $\bar{\theta}$ respectively we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\bar{u}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2) + \underline{\kappa} \|\nabla \bar{\theta}\|_{L^2}^2 \\ & \leq \int_{\Omega} \bar{\theta} \mathbf{e}_2 \cdot \bar{u} dx - \int_{\Omega} (\bar{u} \cdot \nabla u_2) \cdot \bar{u} dx - \int_{\Omega} (\bar{u} \cdot \nabla \theta_2) \bar{\theta} dx \\ & \quad - \int_{\Omega} (\kappa(\theta_1) - \kappa(\theta_2)) \nabla \theta_2 \cdot \nabla \bar{\theta} dx. \end{aligned} \quad (74)$$

For the RHS of (74), we have

$$\begin{aligned}
& \int_{\Omega} \bar{\theta} e_2 \cdot \bar{u} \, d\mathbf{x} - \int_{\Omega} (\bar{u} \cdot \nabla u_2) \cdot \bar{u} \, d\mathbf{x} - \int_{\Omega} (\bar{u} \cdot \nabla \theta_2) \bar{\theta} \, d\mathbf{x} \\
& \leq \|\bar{\theta}\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2 + \|\nabla u_2\|_{L^\infty}^2 \|\bar{u}\|_{L^2}^2 + \|\nabla \theta_2\|_{L^\infty}^2 (\|\bar{\theta}\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2) \\
& \leq C (\|\bar{\theta}\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2), \tag{75}
\end{aligned}$$

and

$$\begin{aligned}
& - \int_{\Omega} (\kappa(\theta_1) - \kappa(\theta_2)) \nabla \theta_2 \cdot \nabla \bar{\theta} \, d\mathbf{x} \\
& \leq \frac{\kappa}{2} \|\nabla \bar{\theta}\|_{L^2}^2 + C \|(\kappa(\theta_1) - \kappa(\theta_2)) \nabla \theta_2\|_{L^2}^2 \\
& \leq \frac{\kappa}{2} \|\nabla \bar{\theta}\|_{L^2}^2 + C \|\nabla \theta_2\|_{L^\infty}^2 \|\kappa(\theta_1) - \kappa(\theta_2)\|_{L^2}^2. \tag{76}
\end{aligned}$$

To estimate $\|\kappa(\theta_1) - \kappa(\theta_2)\|_{L^2}^2$, we have

$$\|\kappa(\theta_1) - \kappa(\theta_2)\|_{L^2}^2 = \left\| \int_{\theta_2}^{\theta_1} \kappa'(s) \, ds \right\|_{L^2}^2 \leq C \|\bar{\theta}\|_{L^2}^2. \tag{77}$$

So we obtained from (74) (by using (75)-(77)) that

$$\frac{1}{2} \frac{d}{dt} (\|\bar{u}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2) + \kappa \|\nabla \bar{\theta}\|_{L^2}^2 \leq \frac{\kappa}{2} \|\nabla \bar{\theta}\|_{L^2}^2 + C (\|\bar{u}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2).$$

Carrying over the term $\frac{\kappa}{2} \|\nabla \bar{\theta}\|_{L^2}^2$ to the LHS, we find

$$\frac{d}{dt} (\|\bar{u}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2) + \kappa \|\nabla \bar{\theta}\|_{L^2}^2 \leq C (\|\bar{u}\|_{L^2}^2 + \|\bar{\theta}\|_{L^2}^2). \tag{78}$$

Applying Gronwall's inequality to (78), we have

$$\|\bar{u}(\cdot, t)\|_{L^2}^2 + \|\bar{\theta}(\cdot, t)\|_{L^2}^2 \leq e^{-Ct} (\|\bar{\theta}_0\|_{L^2}^2 + \|\bar{u}_0\|_{L^2}^2) = 0,$$

for any $t \in [0, T]$. This completes the proof of this theorem. \square

Theorem 3.1 and Theorem 4.1 imply our main result, Theorem 1.1.

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REFERENCES

- [1] J. P. Bourguignon and H. Brezis, Remarks on the Euler equation, *J. Funct. Anal.*, 1975, 341-363.
- [2] J. R. Cannon, and E. DiBenedetto, The initial value problem for the Boussinesq equations with data in L^p . Approximation methods for Navier-Stokes problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979), *Lecture Notes in Math.*, 771, Springer, Berlin, 1980, pp. 129-144,
- [3] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, *Adv. Math.*, 203(2006), 497-513.
- [4] L. C. Evans, *Partial differential equations*, Grad. Stud. Math, 19, Amer. Math. Soc., Providence, RI, 1998.
- [5] T.Y. Hou and C. M. Li, Global well-posedness of the viscous Boussinesq equations, *Discrete Contin. Dynam. Systems* 12 (2005), 112.
- [6] T. Kato, On classical solutions of the two-dimensional non-stationary Euler equation, *Arch. Rational Mech. Anal.*, 25(1967), 188-200.
- [7] S. Klainerman, A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and incompressible limit of compressible fluids, *CPAM*, Vol. XXXIV(1981), 481-524.
- [8] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and quasilinear equations of parabolic type*, American Mathematical Society, 1968.
- [9] M. J. Lai, R. H. Pan and K. Zhao, Initial boundary value problem for 2D viscous Boussinesq equations, *Arch. Rational Mech. Anal.*, 199(2011), 739-760.
- [10] H. P. Li, Some mathematical problems on Boussinesq Equations with nonlinear diffusion, Ph.D. Thesis, Jilin University, 2013.
- [11] P. L. Lions, *Mathematical topics in fluid mechanics*, vols. I and II, Oxford Univ. Press, NewYork, 1996 and 1998.
- [12] Sebastián A. Lorca and José Luiz Boldrini, The initial value problem for a generalized Boussinesq model: regularity and global existence of strong solutions, *Mat. Contemp.* 11(1996), 71-94.
- [13] Sebastián A. Lorca and José Luiz Boldrini, The initial value problem for a generalized Boussinesq model, *Nonlinear Analysis*, 36(1999), 457-480.
- [14] A. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*. Courant Lecture Notes in Mathematics, no. 9. AMS/CIMS, 2003.
- [15] A. Majda and A. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts Appl. Math., 27, Cambridge Univ. Press, Cambridge, 2002.
- [16] H. Morimoto, Nonstationary Boussinesq equations, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 39 (1992), 61-75.

- [17] K. Óeda, On the initial value problem for the heat convection equation of Boussinesq approximation in a time-dependent domain, Proc. Japan Acad. 64, Ser. A (1988), 143-146.
- [18] J. Pedlosky, Geophysical fluid dynamics, Springer-Verlag, NewYork, 1987.
- [19] R. Temam, Navier-Stokes equations, theory and numerical analysis, Stud. Math. Appl., 2, North-Holland, Amsterdam, 1977.
- [20] Y. Sun and Z. Zhang, Global regularity for the initial-boundary value problem of a generalized 2-D Boussinesq system, (2012), Preprint.
- [21] C. Wang and Z. Zhang, Global well-posedness for the 2-D Boussinesq system with the temperature-dependent viscosity and thermal diffusivity, Adv. in Math., 228(2011), 43-62.
- [22] K. Zhao, 2D inviscid heat conductive Boussinesq equations on a bounded domain, Michigan Math. J., 59(2010), 329-352.

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