Darcy's law in one-dimensional isentropic porous medium flow

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Abstract

We study the asymptotic behavior of compressible isentropic flow through porous medium with general L^∞ initial data. The model system is the compressible Euler equation with frictional damping. As $t \to \infty$, the density is conjectured to obey to the well-known porous medium equation and the momentum is expected to be formulated by Darcy's law. Recent progress gives a definte answer to this conjecture for one-dimenstional isentropic flow through the important entropy dissipation principle.

1 Introduction

This note is to survey the recent progress of the program toward the mathematical justification of Darcy law as long time asymptotic limit for compressible isentropic porous medium flow, modeled by the following Cauchy problem of compressible Euler equation with frictional damping,

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = -\alpha \rho u, \\ \rho(x,0) = \rho_0(x), \ u(x,0) = u_0(x). \end{cases} \tag{1}$$

Here ρ, u and $p = \kappa\rho^\gamma$, $\kappa = \frac{(\gamma-1)^2}{4\gamma}$, $(1 < \gamma < 3)$ denotes density, velocity, momentum and pressure, respectively. $\alpha > 0$ is a given positive constant modeling frictional force induced by the medium. We also use momentum $m =: \rho u$ in what follows for convenience. For simplicity, we assume $\alpha = \kappa = \frac{(\gamma-1)^2}{4\gamma}$. Such choice of constants κ and α is purely for convenience, which simplifies the form of the entropy functions we employed below.

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Roughly speaking, because of the external frictional force, the inertial terms in the momentum equation decay to zero faster than other terms so that the pressure gradient force is balanced by the frictional force, which was stated as Darcy law. Therefore, one expects that, as $t \to \infty$, the density is approximated by certain solution of the well-known porous medium equation and the momentum is formulated by Darcy law, as observed in experiments. Therefore, time asymptotically, the system (1) is conjectured to be equivalent to the following decoupled system

$$\begin{cases} \bar{\rho}_t &= (\bar{\rho}^\gamma)_{xx}, \quad \textbf{Porous Medium Equation} \\ \bar{m} &= -(\bar{\rho}^\gamma)_x, \quad \textbf{Darcy Law}. \end{cases} \tag{2}$$

Mathematical study of system (1) dated back to 1970s. Following the pioneer work of Nishida [29], many contributions have been made for this problem. During the early stage of the development, major attentions were paid on the case away from vacuum, system (1) can be transferred to the damped p-system by changing to the Lagrangian coordinates; see [34]. The frictional damping prevents the breaking of waves with small amplitude, leading to the global existence of smooth solutions when initial data is small and smooth [29]. However, waves break down in finite time when the initial derivatives of initial data exceed certain threshold [37]. Therefore, nice theory were developed mainly in the class of small smooth solution away from vacuum. The asymptotic limit has been well justified in such case by Hsiao and Liu in [11] and [12], and further improved by many mathematicians for small smooth or piecewise smooth solutions away from vacuum based on the energy estimates for derivatives; see [10], [13], [14], [15], [16], [28], [30], [31], [32], and [36]. Recently, Dafermos and Pan [7] constructed the global BV solutions to damped p-system and proved the conjecture with sharp decay rates in L^2. In these results, the solutions of damped p-system were shown to converge to the self-similar solutions of the corresponding porous medium equations constructed in [33] since the end-states of the initial density are away from vacuum.

Recently, huge efforts have been put to tackle the large solutions. This is based on the global existence of weak solutions in L^∞ was established by the method of compensated compactness in [8], [9], [23], [38]and [17]. We remark that in this case, the vacuum is allowed in the solution. When a vacuum occurs in the solution, the difficulty of the problem greatly increased mainly due to the interaction of nonlinear convection, lower order dissipation of damping and the resonance due to vacuum.

Definition 1.1 *For any $T > 0$, the bounded measurable functions $(\rho, m)(x,t) \in L^\infty(\mathbf{R} \times [0,T])$ are called entropy solutions of (1), if*

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (\frac{m^2}{\rho} + \kappa\rho^\gamma)_x + \alpha m = 0, \\ \eta_t + q_x + \alpha\eta_m m \le 0, \end{cases} \tag{3}$$

hold in the sense of distributions, where (η, q) is any weak convex entropy–flux pair

$(\eta(\rho,m), q(\rho,m))$ *satisfying*

$$\nabla q = \nabla\eta\nabla f, \quad f = (m, \frac{m^2}{\rho} + \kappa\rho^\gamma)^t, \quad \eta(0,0) = 0. \tag{4}$$

Since the L^∞ weak solution does not have any degree of regularity, the methods for the case away from vacuum are not applicable here. Earlier attempts were made by Huang and Pan [17], where the authors followed the rescaling argument due to Serre and Hsiao [35] and obtained the first justification to the conjecture for vacuum case. In [18], Huang and Pan started to develope a new technique based on the conservation of mass and entropy dissipation analysis to attack this conjecture. They showed that the L^∞ weak entropy solutions with vacuum, selected by the physical entropy-flux pairs, converge strongly in $L^p(R)$ ($p \geq p_0$ for some $p_0 \geq 2$) with decay rates to the similarity solution of the porous medium equation determined uniquely by the end-states and the mass distribution of the initial data provided that the end-states are away from vacuum. This approach seems remarkable since it does not need smallness assumptions on the solutions. Inspired by this result, Huang, Marcati and Pan [19] further studied this problem with finite total mass, where the asymptotic profile is the celebrated Barenblatt's solution of the porous medium equation. These results, valid for some physical flows, depend only on the physcial mechanical energy.

In order to give a further definite answer to the problem, Huang, Pan and Wang devoted further efforts in [20] and [21] to explore the better application of the rich entropy of the isentropic Euler system. In these two papers, they proved that the conjecture is true for most physical flows, that is any entropy L^∞ weak solutions.

In section 2, the self-similar solution and the Barenblatt solution of porous medium equation are reviewed. In section 3, some results on the entropy were collected, and an invariant region result is proved for L^∞ entropy solutions of (1). Finally, in section 4, the main results were presented along with an outline of the proof. In case when one of the end states is away from vacuum, we prove that any L^∞ weak entropy solutions converges to the corresponding self-similar solution of porous medium equation. When the initial total mass is finite, we prove that any L^∞ weak entropy solutions converges to the Barenblatt solution, with a surprising L^1 decay results.

2 Self-similar solutions and Barenblatt solutions

The PME is strictly parabolic if $\rho > 0$ and degenerate if $\rho = 0$. This equation has been well-understood, see [1] and [3].

2.1 Self-similar solutions

Consider the following Cauchy problem for porous medium equation(PME):

$$\begin{cases} \rho_t = (\rho^\gamma)_{xx}, \ \gamma \geq 1, t > 0, \\ \rho(x,0) = \rho_0(x) = \rho_-\chi(x<0) + \rho_+\chi(x>0), \end{cases} \tag{5}$$

where, χ is the characteristic function and $\rho_\pm$ are non-negative constants. In [17], Huang and Pan proved the uniqueness of weak solutions with L^∞ initial data for PME. Since the initial data is self-similar, the uniqueness theorem of [17] implies that ρ itself is self-similar, i.e., $\rho(x,t)=\rho_s(z)$ ($z=\frac{x}{\sqrt{t}}$), which satisfies

$$\begin{cases} (\gamma\bar{\rho}^{\gamma-1}\bar{\rho}_z)_z+\frac{1}{2}z\bar{\rho}_z=0, \\ \bar{\rho}(-\infty)=\rho_-,\ \bar{\rho}(+\infty)=\rho_+. \end{cases} \tag{6}$$

The following two lemmas provide some some properties on ρ_s.

Lemma 2.1 *If $\rho_-\geq\rho_+>0$ then (5) has a unique solution $\rho_s(z)\in C^2$ satisfying the following:*

(1) $\rho_+\leq\rho_s(z)\leq\rho_-$ is monotone decreasing on $\mathbf{R}$.

(2) $(|(\rho_s)_x|,|(\rho_s)_t|)\leq C(t_1)(t^{-\frac{1}{2}},t^{-1})$ for any $t\geq t_1>0$.

Lemma 2.2 *If $\rho_+=0$ and $\rho_->0$, then there is one and only one solution $\rho_s(z)$ to (5). Furthermore, the follows hold.*

- *(1) $0\leq\rho_s(z)\leq\rho_-$ is continuous and monotone decreasing on* $\mathbf{R}$.
- *(2) ${\rho_s}^\gamma(z)$ is smooth on* $\mathbf{R}$.
- *(3) There is a number $b>0$, such that $\rho_s(z)>0$ if $z<b$ and $\rho_s(z)=0$ if $z\geq b$.*
- *(4) $\rho_s(z)$ is smooth if $z<b$.*
- *(5) $({\rho_s}^\gamma)'(z)\to 0$ as $z\to b-0$.*
- *(6) $(|\partial_x({\rho_s}^{\gamma-1})|,|\partial_t({\rho_s}^{\gamma-1})|)\leq C(t_1)(t^{-\frac{1}{2}},t^{-1})$ for any $t\geq t_1>0$.*

2.2 Barenblatt solutions

According to [19], the solutions to (1) with finite total mass should converge in large time to the fundamental solutions of the porous media equation, i.e., the Barenblatt's solutions [2].

Consider

$$\begin{cases} \bar{\rho}_t=(\bar{\rho}^\gamma)_{xx}, \\ \bar{\rho}(-1,x)=M\delta(x), \quad M>0, \end{cases} \tag{7}$$

which admits a unique solution (c.f. [1], [2]) given below

$$\rho_b(x,t)=(t+1)^{-\frac{1}{\gamma+1}}\{(A-B\xi^2)_+\}^{\frac{1}{\gamma-1}}. \tag{8}$$

Here $\xi=x(t+1)^{-\frac{1}{\gamma+1}}$, $(f)_+=\max\{0,f\}$, $B=\frac{\gamma-1}{2\gamma(\gamma+1)}$ and A is determined by

$$2A^{\frac{\gamma+1}{2(\gamma-1)}}B^{-\frac{1}{2}}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{\gamma+1}{\gamma-1}}\,d\theta=M. \tag{9}$$

The following two lemmas summarize some of the properties of ρ_b.

Lemma 2.3 *If $M > 0$ is finite, then there is one and only one solution $\rho_b(x,t)$ to (11). Furthermore, the follows hold.*

- $\rho_b(x,t)$ *is continuous on* $\mathbf{R}$.
- *There is a number* $b = \sqrt{\frac{A}{B}} > 0$, *such that* $\rho_b(x,t) > 0$ *if* $|x| < bt^{\frac{1}{\gamma+1}}$ *and* $\rho_b(x,t) = 0$ *if* $|x| \geq bt^{\frac{1}{\gamma+1}}$.
- $\rho_b(x,t)$ *is smooth if* $|x| < bt^{\frac{1}{\gamma+1}}$.

In terms of the explicit form of ρ_b, it is easy to check the following estimates.

Lemma 2.4 *For ρ_b defined in (8) and $t > 0$, it holds that*

$$\begin{cases} |\rho_b| \leq C(1+t)^{-\frac{1}{\gamma+1}}, \\ |({\rho_b}^{\gamma-1})_x| \leq C(1+t)^{-\frac{\gamma}{\gamma+1}},\ |({\rho_b}^{\gamma-1})_t| \leq C(1+t)^{-\frac{2\gamma}{\gamma+1}}, \\ |({\rho_b}^{\gamma})_x| \leq C(1+t)^{-1},\ |({\rho_b}^{\gamma})_t| \leq C(1+t)^{-\frac{2\gamma+1}{\gamma+1}}. \end{cases} \tag{10}$$

3 Entropy and Invariant Region

First of all, we recall some results on the entropies available for (1). According to [22], all weak entropies of (1) are given by the following formula:

$$\begin{aligned} \eta(\rho,u) &= \int g(\xi)\chi(\xi;\rho,u)\ d\xi = \rho\int_{-1}^{1} g(u+z\rho^\theta)(1-z^2)^\lambda\ dz, \\ q(\rho,m) &= \int g(\xi)(\theta\xi+(1-\theta)u)\chi(\xi;\rho,u)\ d\xi \\ &= \rho\int_{-1}^{1} g(u+z\rho^\theta)(u+\theta z\rho^\theta)(1-z^2)^\lambda\ dz \end{aligned} \tag{11}$$

where $\theta = \frac{\gamma-1}{2}$, $\lambda = \frac{3-\gamma}{2(\gamma-1)}$ and $g(\xi)$ is any smooth function of ξ and

$$\chi(\xi;\rho,u) = (\rho^{\gamma-1} - (\xi-u)^2)_+^\lambda. \tag{12}$$

This remarkable formula can be derived from the entropy equation (4) utilizing the kinetic formulation or by fundamental solution of linear wave equation. We remark that when $g(\xi) = 1$, $\eta(\rho,m) = \rho$; when $g(\xi) = \xi$, $\eta(\rho,m) = m$ and when $g(\xi) = \frac{1}{2}\xi^2$, then

$$\eta_e = \frac{m^2}{2\rho} + \frac{\kappa}{\gamma-1}\rho^\gamma$$

is the mechanical energy.

As the convexity of entropy function is crucial in the definition of admissible weak solutions, the characterization of convexity of entropy functions is important. In our case, the following lemma provides full details in this direction.

Lemma 3.1 *(Lions-Perthame-Tadmor, [22]) Weak entropy $\eta(\rho, m)$ defined in (11) is convex with respect to ρ and m if and only if $g(\xi)$ is a convex function.*

Another important entropy in our context is the one measures $L^{\gamma+1}$ norm in density. Choosing $g(\xi) = |\xi|^{\frac{2\gamma}{\gamma-1}}$, this entropy reads as

$$\tilde{\eta} = \rho \int_{-1}^{1} |u + z\rho^{\theta}|^{\frac{2\gamma}{\gamma-1}} (1 - z^2)^{\lambda} dz. \tag{13}$$

Recall that $B(p, q)$ the Beta function is defined by

$$B(p, q) = \int_{0}^{1} x^{p-1}(1 - x)^{q-1} dx.$$

It is proved in [20] that

Lemma 3.2 *For entropy $\tilde{\eta}$ defined in (13), it holds that*

$$\tilde{\eta} = C_1 \rho^{\gamma+1} + C_2 m^2 + A(\rho, m), \tag{14}$$

where

$$C_1 = \frac{1}{2} B(\frac{\gamma+1}{2(\gamma-1)}, \frac{\gamma+1}{2(\gamma-1)}), \ C_2 = \frac{\gamma(\gamma+1)}{(\gamma-1)^2} B(\frac{\gamma+1}{2(\gamma-1)}, \frac{\gamma+1}{2(\gamma-1)}).$$

Furthermore, $A(\rho, m) \geq 0$, $A_m m \geq 0$, and

$$|A| \leq C\rho |u|^3 (|u|^{\frac{3-\gamma}{\gamma-1}} + \rho^{1-\theta}).$$

As a direct application of entropy analysis, we will present an invariant region theorem for L^{∞} weak entropy solutions of (1). Choosing $g(\xi) = g_k(\xi) = e^{k\xi^2}$ in (11), for positive parameter $k > 0$, the corresponding entropy $\eta_k \geq 0$ is clearly convex. Using this sequence of entropies, [20] proved the following invariant region theorem, which confirms that any L^{∞} weak entropy solutions to (1) will stay inside the physical region

$$0 \leq \rho(x, t) \leq C, \ |m(x, t)| \leq C\rho(x, t),$$

if the initial data does so. More precisely, the theorem read as

Theorem 3.1 *(Huang, Pan, Wang, [20]) Suppose $(\rho_0, u_0)(x) \in L^{\infty}(\mathbf{R})$ such that*

$$0 \leq \rho_0(x) \leq C, \ |m_0(x)| \leq C\rho_0(x).$$

Let $(\rho, u) \in L^{\infty}(R \times [0, T])$ be an L^{∞} weak entropy solution of the system (1) with $\gamma > 1$. Then (ρ, m) satisfies

$$0 \leq \rho(x, t) \leq C, \ |m(x, t)| \leq C\rho(x, t), \tag{15}$$

where the constant C depends solely on the initial data.

4 Large time behavior

In this section, we will report the main results obtained in [20] and [21] for the large time asymptotic behavior of general L^∞ weak entropy solutions of (1). For this purpose, we suppose that (ρ, m) is a L^∞ weak entropy solution of (1) such that

$$0 \le \rho_0(x) \le C,\ \ |m_0(x)| \le C\rho_0(x). \tag{16}$$

For the long time profile, we will also denote by ρ_s the self-similar solution of PME, by ρ_b the Barenblatt solution of PME. Due to the limitation of the length, we will only outline the proof for the case with finite total mass.

4.1 Infinite total mass

In this case, one assumes that

$$\lim_{x\to\pm\infty}(\rho_0(x), m_0(x)) = (\rho_\pm, m_\pm). \tag{17}$$

When the initial end stats of density has a non-vacuum state, say $\rho_- > 0$, the results in [21] confirms that (ρ, m) converges to the corresponding self-similar profiles ρ_s in large time. With the procedure introduced in [11] and [12], one could assume that $m_- = m_+ = 0$ and

$$\int_{-\infty}^{\infty} (\rho_0(x) - \rho_s(x, 0))\ dx = 0. \tag{18}$$

A convenient variable y is thus introduced

$$y = -\int_{-\infty}^{x} (\rho - \rho_s)(r, t)\ dr. \tag{19}$$

The following two theorems are proved in [21].

Theorem 4.1 *Assume* $1 < \gamma \le 2$, $\rho_- > \rho_+ = 0$ *and* $m_\pm = 0$*. Suppose* (ρ, m) *is a* L^∞ *weak entropy solution of (1) satisfies (16) and (17). Let* ρ_s *be the selfsimilar solution of (1.3) with* $\rho_s(\pm\infty) = \rho_\pm$ *and* $\bar{m} = -P(\rho_s)_x$*. If* $y \in H^1(\mathbf{R})$*, then there exists positive constant* C *such that, for any* $\varepsilon > 0$,

$$\|\rho - \rho_s\|_{L^{\gamma+1}}^{\gamma+1} \le C(1+t)^{-\frac{1}{4}+\varepsilon}. \tag{20}$$

Theorem 4.2 *Assume* $1 < \gamma < 3$, $\rho_- \ge \rho_+ > 0$*, and* $m_- = m_+ = 0$*. Suppose* (ρ, m) *is a* L^∞ *weak entropy solution of (1) satisfies (16) and (17). Let* ρ_s *be the selfsimilar solution of (1.3) with* $\rho_s(\pm\infty) = \rho_\pm$ *and* $\bar{m} = -P(\rho_s)_x$*. If* $y_0(x) \in H^1$*, then there exist constants* $C > 0$ *and* $0 < \beta < \frac{1}{2}$ *independent of time such that*

$$\|(\rho(x,t) - \rho_s(x,t), m(x,t) - \bar{m}(x,t))\|_{L^p} \le C(1+t)^{-\beta/p},\quad 2 \le p < \infty. \tag{21}$$

4.2 Finite total mass

In this case, one assumes that

$$\int_{-\infty}^{+\infty} \rho(x,t)\ dx = \int_{-\infty}^{+\infty} \rho_b(x,t)\ dx = \int_{-\infty}^{+\infty} \rho_0(x)\ dx = M < \infty. \tag{22}$$

The following Theorem, proved in [20], shows that $\rho(x,t)$ converges to the corresponding Barenblatt solution ρ_b with the same total mass. The decay rates in both energy norm ($L^{\gamma+1}$) and mass norm (L^1) are obtained.

Theorem 4.3 *Suppose $\rho_0(x) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $u_0(x) \in L^\infty(\mathbf{R})$ and*

$$M = \int_{-\infty}^{\infty} \rho_0(x)\ dx > 0.$$

Let $1 < \gamma < 3$ and (ρ, m) be an L^∞ entropy solution of the Cauchy problem (1). Let ρ_b be the Barenblatt's solution of porous medium equation (2) with mass M and $\bar{m} = -({\rho_b}^\gamma)_x$. Define

$$y = -\int_{-\infty}^{x} (\rho - \rho_b)(r,t)\ dr.$$

If $y(x,0) \in L^2(\mathbf{R})$, then for any $\varepsilon > 0$ and $t > 0$,

$$\begin{aligned} &\|(\rho-\rho_b)(\cdot,t)\|_{L^{\gamma+1}}^{\gamma+1} \le C(1+t)^{-1+\frac{1}{2(\gamma+1)}+\varepsilon}, \\ &\|(\rho-\rho_b)(\cdot,t)\|_{L^1} \le C(1+t)^{-\frac{1}{4(\gamma+1)}+\varepsilon}. \end{aligned} \tag{23}$$

4.3 Outline of the proof to Theorem 4.3

In order to control the nonlinearity and singularity near vacuum, the following lemma is proved in [20].

Lemma 4.1 *If $0 \le \rho, \bar{\rho} \le C$, there are two constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\begin{cases} c_1(\rho^{\gamma-1}+\bar{\rho}^{\gamma-1})(\rho-\bar{\rho})^2 & \le \rho^{\gamma+1} - \bar{\rho}^{\gamma+1} - (\gamma+1)\bar{\rho}^\gamma(\rho-\bar{\rho}) \\ & \le c_2(\rho^{\gamma-1}+\bar{\rho}^{\gamma-1})(\rho-\bar{\rho})^2 \\ c_1(\rho^{\gamma-1}+\bar{\rho}^{\gamma-1})(\rho-\bar{\rho})^2 & \le (\rho^\gamma - \bar{\rho}^\gamma)(\rho-\bar{\rho}) \le c_2(\rho^{\gamma-1}+\bar{\rho}^{\gamma-1})(\rho-\bar{\rho})^2. \end{cases} \tag{24}$$

Using the notion y, the momentum equation is equivalent to

$$y_{tt} + (\frac{m^2}{\rho})_x + \kappa(\rho^\gamma - \bar{\rho}^\gamma)_x + \kappa y_t = -\bar{m}_t. \tag{25}$$

By standard energy method, we first obtain the following estimate

Lemma 4.2 *Let the conditions of Theorem 4.3 be satisfied, it holds*

$$\begin{aligned}&\int_{-\infty}^{+\infty}(y_ty+\frac{\kappa}{2}y^2)\ dx+\int_0^t\int_{-\infty}^{+\infty}\kappa(\rho^\gamma-\bar{\rho}^\gamma)(\rho-\bar{\rho})\ dxd\tau\\ \le\ & C+\int_0^t\int_{-\infty}^{+\infty}y_t^2\ dxd\tau+\int_0^t\int_{-\infty}^{+\infty}\frac{m^2}{\rho}y_x\ dxd\tau.\end{aligned}\tag{26}$$

In the second step, we use mechanical energy

$$\eta_e=\frac{m^2}{2\rho}+\frac{\kappa}{\gamma-1}\rho^\gamma,$$

and q_e the corresponding flux in the entropy inequality,

$$\eta_{et}+q_{ex}+\kappa\frac{m^2}{\rho}\le 0,\tag{27}$$

to prove the following estimate

Lemma 4.3 *Under the conditions of Theorem 4.3, for any $t>0$, it holds that*

$$\begin{aligned}&\int_{-\infty}^{+\infty}(\rho^\gamma+y_t^2+\frac{m^2}{\rho}+y^2)\ dx+\int_0^t\int_{-\infty}^{+\infty}(\frac{m^2}{\rho}+y_t^2)\ dxd\tau\\ &+\int_0^t\int_{-\infty}^{+\infty}(\rho^\gamma-{\rho_b}^\gamma)(\rho-\rho_b)\ dxd\tau\le C.\end{aligned}\tag{28}$$

In the third step, we employ the entropy-entropy flux pair $(\tilde{\eta},\tilde{q})$, which induces

$$\eta_*=\tilde{\eta}-C_1{\rho_b}^{\gamma+1}-C_1(\gamma+1){\rho_b}^\gamma(\rho-\rho_b).\tag{29}$$

The entropy inequality implies

$$\eta_{*t}+(C_1{\rho_b}^{\gamma+1}+C_1(\gamma+1){\rho_b}^\gamma(\rho-\rho_b))_t+\tilde{q}_x+2\kappa C_2m^2+\kappa A_mm\le 0,\tag{30}$$

which further implies that

$$\eta_{*t}-2\kappa C_2({\rho_b}_x^\gamma y)_t+2\kappa C_2(m-\bar{m})^2+4\kappa C_2({\rho_b}^\gamma)_t(\rho-\rho_b)+\kappa A_mm+(\cdots)_x\le 0.\tag{31}$$

For any small positive constant $\varepsilon>0$, we define

$$\mu(\varepsilon)=1-\frac{1}{2(\gamma+1)}-\varepsilon.$$

Multiplying (31) by $(1+t)^{\mu(\varepsilon)}$, integrating the result on $\mathbf{R}\times[0,t]$ and using Lemma 4.4, we shall reach, after some computation, that

$$\int_{-\infty}^{+\infty}\eta_*\ dx\le C(1+t)^{-\mu(\varepsilon)}.\tag{32}$$

Therefore we proved the following lemma,

Lemma 4.4 *Under the conditions of Theorem 4.3, it holds for any $t > 0$ that*

$$\|(m-\bar{m})(\cdot,t)\|_{L^2}^2 + \|(\rho-\rho_b)(\cdot,t)\|_{L^{\gamma+1}}^{\gamma+1} + \int_{-\infty}^{+\infty} (\rho^{\gamma-1} + {\rho_b}^{\gamma-1})(\rho-\rho_b)^2 \, dx \leq C(1+t)^{\frac{1}{2(\gamma+1)}+\varepsilon-1}, \tag{33}$$

$$\int_0^t (1+\tau)^{1-\frac{1}{2(\gamma+1)}-\varepsilon} \|(m-\bar{m})(\cdot,\tau)\|_{L^2}^2 \, d\tau \leq C,$$

for any positive constant ε.

This crucial sharper decay rates in $L^{\gamma+1}$, together with the following key observation will lead to the decay in L^1.

Lemma 4.5 *If $\rho \geq 0$ and $\bar{\rho} \geq 0$ have the same total mass M, then for any $t > 0$,*

$$\int_{-\infty}^{+\infty} |\rho-\bar{\rho}|(x,t)dx \leq 2\int_{\bar{\rho}>0} |\rho-\bar{\rho}|(x,t)dx. \tag{34}$$

Now, it is straightforward to combine Lemma 4.4 and Lemma 4.5 to prove the following decay estimates.

Lemma 4.6 *Assume the conditions in Theorem 1.2 are satisfied, then*

$$\|\rho-\rho_b\|_{L^1} \leq C(1+t)^{-\frac{1}{4(\gamma+1)}+\varepsilon}, \ \forall t > 0,$$

for any $\varepsilon > 0$.

We thus conclude the proof of Theorem 4.3 from Lemmas 4.4 and 4.6.

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