**ARMA manuscript No.**
(will be inserted by the editor)

# $L^1$ convergence to the Barenblatt solution for compressible Euler equations with damping

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**Abstract**

We study the asymptotic behavior of compressible isentropic flow through porous medium when the initial mass is finite. The model system is the compressible Euler equation with frictional damping. As $t \to \infty$, the density is conjectured to obey to the well-known porous medium equation and the momentum is expected to be formulated by Darcy's law. In this paper, we prove that any $L^\infty$ weak entropy solution to the Cauchy problem of damped Euler equations with finite initial mass converges strongly in the natural $L^1$ topology with decay rates to the Barenblatt's profile of porous medium equation. The density function tends to the Barenblatt's solution of porous medium equation while the momentum is described by the Darcy's law. The results are achieved through a comprehensive entropy analysis, capturing the dissipative character of the problem.

## 1. Introduction

This paper is the continuation of the program toward the mathematical justification of Darcy law as long time asymptotic limit for compressible isentropic porous medium flow, modeled by the following Cauchy problem of compressible Euler equation with frictional damping,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = -\alpha \rho u, \\ \rho(x,0) = \rho_0(x), \ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Here $\rho$, $u$ and $p = \kappa \rho^\gamma$, $\kappa = \frac{(\gamma-1)^2}{4\gamma}, (1 < \gamma < 3)$ denotes density, velocity, momentum and pressure, respectively. $\alpha > 0$ is a given positive constant modeling frictional force induced by the medium. We also use momentum $m =: \rho u$ in what follows for convenience. For simplicity, we assume $\alpha = \kappa = \frac{(\gamma-1)^2}{4\gamma}$. Such choice of constants $\kappa$ and $\alpha$ is purely for convenience, which simplifies the form of the entropy functions we employed below.

Because of the dissipative nature of frictional force, it is natural to expect the inertial terms in the momentum equation decay to zero faster than other terms so that the pressure gradient force is balanced by the frictional force, which was stated as Darcy law. In other words, as $t \to \infty$, the density is conjectured to obey to the well-known porous medium equation and the momentum is expected to be formulated by Darcy law, observed in experiments. Therefore, time asymptotically, the system (1.1) is conjectured to be equivalent to the following decoupled system

$$\begin{cases} \bar{\rho}_t &= (\bar{\rho}^\gamma)_{xx}, \quad \textbf{Porous Medium Equation} \\ \bar{m} &= -(\bar{\rho}^\gamma)_x, \quad \textbf{Darcy Law}. \end{cases} \tag{1.2}$$

The particular emphasis here is when the initial total mass is finite allowing vacuum states in the solutions initially, namely,

$$\int_{-\infty}^{+\infty} \rho(x,t)\,dx = \int_{-\infty}^{+\infty} \bar{\rho}(x,t)\,dx = \int_{-\infty}^{+\infty} \rho_0(x)\,dx = M < \infty,$$

in view of the mass conservation. We shall prove in this paper the $L^1$ convergence from any $L^\infty$ entropy weak solution of the compressible Euler equations to the Barenblatt solution [2] of the corresponding porous medium equation carrying finite total mass. The first evidence toward this expectation was hinted in the inspiring paper of Liu [26] through an interesting explicit solution which behaves as the Barenblatt's solution of porous medium equation. Recent evidence was provided by Huang, Marcati and Pan [21] for $L^\infty$ entropy weak solution for $\gamma \in (\frac{1+\sqrt{5}}{2}, \sqrt{2}+1)$ measured in the energy norm of porous medium equation. The purpose of current paper is to finally provide a complete treatment to this case when $\gamma \in (1,3)$, with measurements in both $L^1$ (mass norm) and $L^{\gamma+1}$ (energy norm) and thus in $L^p$ for $1 \le p \le \gamma+1$.

Mathematical study of system (1.1) dated back to 1970s. Following the pioneer work of Nishida [33], many contributions have been made for this problem. In the case away from vacuum, system (1.1) can be transferred to the damped $p$-system by changing to the Lagrangian coordinates; see [43]. The frictional damping prevents the breaking of waves with small amplitude, leading to the global existence of smooth solutions when initial data is small and smooth [33]. However, waves break down in finite time when the initial derivatives of initial data exceed certain threshold [45]. The global existence of weak solutions in $L^p$ was established by the method of compensated compactness in [10], [11], [25] and [46]. The global BV solutions were proved in [27], [7] and [9]. The conjecture mentioned above has been first justified by Hsiao and Liu in [13] and [14], and further improved by many mathematicians for small smooth or piecewise smooth solutions away from vacuum based on the energy estimates for derivatives; see [12], [15], [16], [17], [18], [30], [34], [35], [36], and [44]. Recently, Dafermos and Pan [9] constructed the global BV solutions to damped $p$-system and proved the conjecture with sharp decay rates in $L^2$. In these results, the solutions of damped $p$-system were shown to converge to the self-similar solutions of the corresponding porous medium equations constructed in [40] since the end-states of the initial density are away from vacuum.

When a vacuum occurs in the solution, the difficulty of the problem greatly increased mainly due to the interaction of nonlinear convection, lower order dissipation of damping and the resonance due to vacuum. It is known that the nonlinearity is the reason for shock formation in a hyperbolic system. For hyperbolic conservation laws, the self-similarity is an important feature in constructing fundamental Riemann solutions and in describing the large time behaviors of solutions. Although the damping provides weak dissipation, it breaks the self-similarity of the system which is crucial for the large weak solutions. Another difficulty is due to the resonance near vacuum which develops a new singularity; see [28] and [29]. Due to this new singularity, it is very difficult to obtain the solutions with any degree of regularity. This makes (1.1) difficult to understand analytically and makes the construction of effective numerical methods for computing solutions a highly non-trivial problem. Indeed, the only global weak solution with vacuum is constructed in $L^\infty$ space by using the method of compensated compactness; see Ding, Chen and Luo [10] for $1 < \gamma \leq \frac{5}{3}$ and Huang and Pan [19] for $1 \leq \gamma < 3$. Thus, to study the large time behavior of solution of (1.1) with vacuum, it is suitable to consider the $L^\infty$ weak solution.

**Definition 1.1.** For any $T > 0$, the bounded measurable functions $(\rho,m)(x,t) \in L^\infty(\mathbf{R} \times [0,T])$ are called entropy solutions of (1.1), if

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (\frac{m^2}{\rho} + \kappa\rho^\gamma)_x + \alpha m = 0, \\ \eta_t + q_x + \alpha\eta_m m \leq 0, \end{cases} \tag{1.3}$$

hold in the sense of distributions, where $(\eta,q)$ is any weak convex entropy–flux pair $(\eta(\rho,m),q(\rho,m))$ satisfying

$$\nabla q = \nabla\eta\nabla f, \;\; f = (m, \frac{m^2}{\rho} + \kappa\rho^\gamma)^t, \;\; \eta(0,0) = 0. \tag{1.4}$$

As the $L^\infty$ weak solution does not have any degree of regularity, the methods for the case away from vacuum are not applicable here. Earlier attempts were made by Huang and Pan [19], where the authors followed the rescaling argument due to Serre and Hsiao [42] and obtained the first justification to the conjecture for vacuum case. It showed that the density in the $L^\infty$ weak entropy solutions of (1.1) converges to the similarity solution of porous medium equation along the level curve of the diffusive similarity profiles provided that one of the initial end-states is nonzero. The long time behavior of the momentum is not known however. This is far from satisfactory. In [20], Huang and Pan developed a new technique based on the conservation of mass and entropy analysis to attack this conjecture. They showed that the $L^\infty$ weak entropy solutions with vacuum, selected by the physical entropy-flux pairs, converge strongly in $L^p(R)$ ($p \geq p_0$ for some $p_0 \geq 2$) with decay rates to the similarity solution of the porous medium equation determined uniquely by the end-states and the mass distribution of the initial data, provided that the end-states are away from vacuum. This approach seems remarkable since it does not need smallness assumptions on the solutions. Inspired by this result, Huang, Marcati and Pan [21] further studied this problem with finite total mass, where the asymptotic profile is the celebrated Barenblatt's solution of the porous medium equation. More precisely, this result is cited below:

**Theorem** (Huang-Marcati-Pan, [21]) Suppose $0 \le \rho_0(x) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ and

$$0 < M = \int_{-\infty}^{+\infty} \rho_0(x)\, dx < \infty.$$

Let $(\rho, m)$ be an $L^\infty$ entropy solution of the Cauchy problem (1.1), satisfying the following estimates

$$0 \le \rho(x,t) \le C,\ |m(x,t)| \le C\rho(x,t), \tag{1.5}$$

where the constant $C$ is independent of $t$. Let $\bar\rho$ be the Barenblatt's solution of porous medium equation (1.2) with mass $M$ and $\bar m = -(\bar\rho^\gamma)_x$ which satisfies

$$\begin{aligned} &\|\bar\rho\|_{L^2}^2 \le C(1+t)^{-\frac{1}{\gamma+1}},\\ &\|\bar\rho\|_{L^\gamma}^\gamma \le C(1+t)^{-\frac{\gamma-1}{\gamma+1}}. \end{aligned} \tag{1.6}$$

Define

$$y = -\int_{-\infty}^{x} (\rho - \bar\rho)(r,t)\, dr.$$

If $y(x,0) \in L^2(\mathbf{R})$, then there exist positive constants $k_1 = \min\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{\gamma-1}{\gamma}\}$, $k_2 = \min\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{1}{\gamma}\}$ and $C$ such that for any $\varepsilon > 0$,

$$\begin{aligned} &\|(\rho-\bar\rho)(x,t)\|_{L^2}^2 \le C(1+t)^{-k_1+\varepsilon}, \text{ if } 1 < \gamma \le 2,\\ &\|(\rho-\bar\rho)(x,t)\|_{L^\gamma}^\gamma \le C(1+t)^{-k_2+\varepsilon}, \text{ if } \gamma \ge 2. \end{aligned} \tag{1.7}$$

Furthermore,

$$\begin{aligned} &k_1 > \tfrac{1}{\gamma+1}, \text{ if } \tfrac{1+\sqrt5}{2} < \gamma \le 2,\\ &k_2 > \tfrac{\gamma-1}{\gamma+1}, \text{ if } 2 \le \gamma < 1+\sqrt2. \end{aligned} \tag{1.8}$$

**Remark 1.1.** Since Barenblatt's solution $\bar\rho$ decays itself, it is necessary to compare the decay rate of $\bar\rho$ with that of $\rho - \bar\rho$. This theorem shows that $\|\rho-\bar\rho\|_{L^2}$ decays faster than $\|\bar\rho\|_{L^2}$ when $\frac{1+\sqrt5}{2} < \gamma \le 2$ and $\|\rho-\bar\rho\|_{L^\gamma}$ decays faster than $\|\bar\rho\|_{L^\gamma}$ when $2 \le \gamma < 1+\sqrt2$. Thus this theorem states that any $L^\infty$ entropy weak solutions must converge to the Barenblatt's solution of PME with the same mass when $\gamma \in (\frac{1+\sqrt5}{2}, 1+\sqrt2)$.

**Remark 1.2.** The entropy dissipation method introduced in [20] and [21] is an effective approach in proving the large time asymptotic behavior for $L^\infty$ weak entropy solutions for hyperbolic conservation laws with dissipation. Further application of such approach can be found in [39] for an initial boundary value problem and in [22] for the Euler-Poisson system modeling semi-conductor device.

Although this is the first result providing the convergence of $L^\infty$ weak entropy solutions of (1.1) to the Barenblatt's solution of (1.2), the result itself is not definitive for several reasons. One may thus ask the following questions:

- **Question 1:** Is it possible to remove the assumption (1.5)? The assumption (1.5) is essential in the proof of [21]. This assumption is quite reasonable since the solutions obtained in [10] and [19] do satisfy this condition where the invariant region theory is applied to the viscosity regularized system. On the other hand, there is no uniqueness theory available for the $L^\infty$ weak entropy solutions of (1.1), it is not clear whether there is any uniformly bounded entropy weak solutions of (1.1) that does not satisfy (1.5) with uniform constant $C$. Therefore, it is natural to ask this question.
- **Question 2:** Is it possible to prove the convergence for any $\gamma \in (1,3)$ not only on the interval $(\frac{1+\sqrt{5}}{2}, 1+\sqrt{2})$ as in the last Theorem? Although the interval $(\frac{1+\sqrt{5}}{2}, 1+\sqrt{2})$ contains some physical cases, most physical gases live in the larger interval $(1,3)$. This generalization is thus important from physical point of view.
- **Question 3:** Is it possible to prove decay of $\|\rho-\bar{\rho}\|_{L^1}$? Since the compressible Euler equation is conserved, it is natural to measure the difference between $\rho$ and the Barenblatt solution $\bar{\rho}$ in $L^1$ space. Furthermore, the $L^1$ norm does not decay for either $\rho$ or $\bar{\rho}$, the $L^1$ decay is very convincing. Therefore, the last question is whether it is possible to obtain an $L^1$ convergence result.

In this paper, we will address these three questions listed above and give definite answers to them. After a quick review of some information on Barenblatt's solution in Section 2, we will first prove in Section 3 the following invariant region theory for $L^\infty$ weak entropy solution to (1.1).

**Theorem 1.1.** *Suppose that* $(\rho_0,u_0)(x) \in L^\infty(\mathbf{R})$ *satisfies*

$$0 \le \rho_0(x) \le C, \ |m_0(x)| \le C\rho_0(x).$$

*Let* $(\rho,u) \in L^\infty(R\times[0,T])$ *be an* $L^\infty$ *weak entropy solution of the system (1.1) with* $\gamma > 1$. *Then* $(\rho,m)$ *satisfies*

$$0 \le \rho(x,t) \le C, \ |m(x,t)| \le C\rho(x,t), \tag{1.9}$$

*where the constant C depends solely on the initial data.*

**Remark 1.3.** Theorem 1.1 is valid for $L^\infty$ weak entropy solutions to the homogeneous compressible Euler systems, this is quite clear from the proof in section 3 below. This theorem is an invariant region theorem for the weak solutions to compressible Euler equations with damping. In the invariant region theorem [6], the derivative property of solution is essential. But here it is impossible to use any derivative properties due to the lack of regularity. Theorem 1.1 is proved by choosing infinitely many convex weak entropies. This idea dated back to Dafermos [8] and Serre [41]. Similar idea appears in [24] and [38] for homogeneous two by two system including compressible Euler equations and elastodynamics. The distinct feature in our proof for this thoerem is the choice of entropies designed for the dissipative source term.

In Section 4, we will pursue the sharper decay rates of the density to the Barenblatt's solution, which will finally give the decay rates in $L^1$ distance. These are stated in the following Theorem.

**Theorem 1.2.** *Suppose $\rho_0(x) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $u_0(x) \in L^\infty(\mathbf{R})$ and*

$$M = \int_{-\infty}^{\infty} \rho_0(x)\, dx > 0.$$

*Let $1 < \gamma < 3$ and $(\rho, m)$ be an $L^\infty$ entropy solution of the Cauchy problem (1.1). Let $\bar{\rho}$ be the Barenblatt's solution of porous medium equation (1.2) with mass $M$ and $\bar{m} = -(\bar{\rho}^\gamma)_x$. Define*

$$y = -\int_{-\infty}^{x} (\rho - \bar{\rho})(r,t)\, dr.$$

*If $y(x,0) \in L^2(\mathbf{R})$, then for any $\varepsilon > 0$ and $t > 0$,*

$$\begin{aligned} &\|(\rho-\bar{\rho})(\cdot,t)\|_{L^{\gamma+1}}^{\gamma+1} \le C(1+t)^{-1+\frac{1}{2(\gamma+1)}+\varepsilon}, \\ &\|(\rho-\bar{\rho})(\cdot,t)\|_{L^1} \le C(1+t)^{-\frac{1}{4(\gamma+1)}+\varepsilon}. \end{aligned} \tag{1.10}$$

**Remark 1.4.** Due to the mass conservation law, it is natural to use $L^1$ norm to measure the difference between the entropy solution $\rho$ and Barenblatt's solution $\bar{\rho}$. (1.10) implies that any $L^\infty$ entropy solutions must converge to the Barenblatt's solution of PME with the same mass for $1 < \gamma < 3$. This is becasue the $L^1$ norm is conserved for both $\rho$ and $\bar{\rho}$. Furthermore, we remark that the $L^{\gamma+1}$ norm of $\rho - \bar{\rho}$ decays faster than $\bar{\rho}$, since for the latter one has from Lemma 2.2 below that

$$\|\bar{\rho}(\cdot,t)\|_{L^{\gamma+1}}^{\gamma+1} \le C(1+t)^{-\frac{\gamma}{\gamma+1}}$$

which decays slower than the rate in (1.10) by $(1+t)^{\frac{1}{2(\gamma+1)}}$ roughly.

We now make some comments on the new ideas and approaches in this paper, based on intensive entropy analysis. In the proof of Theorem 1.1, infinitely many convex entropy functions are used. In the proof of Theorem 1.2, we first follow the method of [21] to pave the road. With the help of a sharp estimate on the degeneracy of $p(\rho)$ at vacuum proved in Lemma 3.1, a carefully chosen entropy is applied to obtain a much sharper decay in $L^{\gamma+1}$ norm of the density, which is missing in [21]. This much faster decay rate guaranteed that $\|\rho - \bar{\rho}\|_{L^{\gamma+1}}$ decays faster than $\|\bar{\rho}\|_{L^{\gamma+1}}$ for any $\gamma \in (1,3)$. Finally, this refined estimate, together with a key observation on the distribution of $\|\rho - \bar{\rho}\|_{L^1}$ over the support of $\bar{\rho}$, leads to the $L^1$ decay rates in Theorem 1.2.

In the last section, we further discuss a Barenblatt type solution of (1.1) with a particular initial data, constructed by T. Liu in [26]. This interesting solution, we call it Liu's solution, behaves just like Barenblatt's solution and thus could serve as the large time asymptotic ansatz for solutions of (1.1) with finite total mass. Theorem 1.2, together with the results in [26], implies the $L^1$ decay to Liu's solution as well. The reason for the choice of Barenblatt's solution is simply because of its explicit form which is not available for Liu's solution. Furthermore, the decay rates in Theorem 1.2 might not be optimal, some explanations will be presented there at the end of the paper.

## 2. Barenblatt's solutions

According to [21], the solutions to (1) with finite total mass should converge in large time to the fundamental solutions of the porous media equation, i.e., the Barenblatt's solutions [2]. In this section, we provide some background information on the Barenblatt's solutions.

Consider

$$\begin{cases} \bar{\rho}_t = (\bar{\rho}^{\gamma})_{xx}, \\ \bar{\rho}(-1,x) = M\delta(x), \quad M > 0, \end{cases} \tag{2.1}$$

which admits a unique solution (c.f. [1], [2]) given below

$$\bar{\rho}(x,t) = (t+1)^{-\frac{1}{\gamma+1}}\{(A - B\xi^2)_+\}^{\frac{1}{\gamma-1}}. \tag{2.2}$$

Here $\xi = x(t+1)^{-\frac{1}{\gamma+1}}$, $(f)_+ = \max\{0,f\}$, $B = \frac{\gamma-1}{2\gamma(\gamma+1)}$ and $A$ is determined by

$$2A^{\frac{\gamma+1}{2(\gamma-1)}}B^{-\frac{1}{2}}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{\gamma+1}{\gamma-1}}\,d\theta = M. \tag{2.3}$$

Due to the degeneracy at vacuum, the derivatives of $\bar{\rho}$ is not continuous across the interface between the gas and vacuum. Instead, $\bar{\rho}$ is a weak solution to (2.1) such that

$$\int_{-\infty}^{+\infty}\bar{\rho}\,dx = M, \tag{2.4}$$

and

$$\bar{\rho} = 0, \quad \text{if} \quad |\xi| \geq \sqrt{\frac{A}{B}}. \tag{2.5}$$

Hence, for any finite time $T > 0$, $\bar{\rho}$ has compact support. This is the property of finite speed of propagation for porous medium equation. For the definition of the weak solution to (2.1), we refer to [1], [2], [3], [23] and [31]. Kamin proved in [23] that (2.1) admits at most one solution. Here, we addressed the initial data at $t = -1$ to avoid the singularity at $t = 0$. Thus, we have the following lemmas from (2.2)–(2.5).

**Lemma 2.1.** *If $M > 0$ is finite, then there is one and only one solution $\bar{\rho}(x,t)$ to (2.1). Furthermore, the follows hold.*

– *$\bar\rho(x,t)$ is continuous on* $\mathbf{R}$.
– *There is a number $b=\sqrt{\frac{A}{B}}>0$, such that $\bar\rho(x,t)>0$ if $|x|<bt^{\frac{1}{\gamma+1}}$; and $\bar\rho(x,t)=0$ if $|x|\ge bt^{\frac{1}{\gamma+1}}$.*
– *$\bar\rho(x,t)$ is smooth if $|x|<bt^{\frac{1}{\gamma+1}}$.*

In terms of the explicit form of $\bar\rho$, it is easy to check the following estimates.

**Lemma 2.2.** *For $\bar\rho$ defined in (2.2) and $t>0$, it holds that*

$$\begin{cases} |\bar\rho|\le C(1+t)^{-\frac{1}{\gamma+1}}, \\ |(\bar\rho^{\gamma-1})_x|\le C(1+t)^{-\frac{\gamma}{\gamma+1}},\ |(\bar\rho^{\gamma-1})_t|\le C(1+t)^{-\frac{2\gamma}{\gamma+1}}, \\ |(\bar\rho^{\gamma})_x|\le C(1+t)^{-1},\ |(\bar\rho^{\gamma})_t|\le C(1+t)^{-\frac{2\gamma+1}{\gamma+1}}, \end{cases} \tag{2.6}$$

*and*

$$\begin{cases} \int_{-\infty}^{+\infty}\bar\rho^p\,dx\le C(1+t)^{-\frac{p-1}{\gamma+1}},\ \forall p\ge 1, \\ \int_{-\infty}^{+\infty}(\bar\rho^{\gamma-1})_x^2\,dx\le C(1+t)^{-\frac{2\gamma-1}{\gamma+1}},\ \int_{-\infty}^{+\infty}(\bar\rho^{\gamma-1})_t^2\,dx\le C(1+t)^{-\frac{4\gamma-1}{\gamma+1}}, \\ \int_{-\infty}^{+\infty}(\bar\rho^{\gamma})_x^2\,dx\le C(1+t)^{-\frac{2\gamma+1}{\gamma+1}},\ \int_{-\infty}^{+\infty}(\bar\rho^{\gamma})_t^2\,dx\le C(1+t)^{-\frac{4\gamma+1}{\gamma+1}}. \end{cases} \tag{2.7}$$

## 3. Invariant Region for weak solutions

In this section, we shall show a proof to Theorem 1.1, which serves as invariant region theory [6] for $L^\infty$ weak entropy solutions to (1.1). It confirms that any $L^\infty$ weak entropy solutions to (1.1) will stay inside the physical region

$$0\le\rho(x,t)\le C,\ |m(x,t)|\le C\rho(x,t),$$

if the initial data does so.

To proceed, we first recall some results on the entropies available for (1.1). According to [24], all weak entropies of (1.1) are given by the following formula:

$$\begin{aligned} \eta(\rho,u)&=\int g(\xi)\chi(\xi;\rho,u)\,d\xi=\rho\int_{-1}^{1}g(u+z\rho^\theta)(1-z^2)^\lambda\,dz, \\ q(\rho,m)&=\int g(\xi)(\theta\xi+(1-\theta)u)\chi(\xi;\rho,u)\,d\xi \\ &=\rho\int_{-1}^{1}g(u+z\rho^\theta)(u+\theta z\rho^\theta)(1-z^2)^\lambda\,dz \end{aligned} \tag{3.1}$$

where $\theta=\frac{\gamma-1}{2}$, $\lambda=\frac{3-\gamma}{2(\gamma-1)}$, $g(\xi)$ is any smooth function of $\xi$, and

$$\chi(\xi;\rho,u)=(\rho^{\gamma-1}-(\xi-u)^2)_+^\lambda. \tag{3.2}$$

This remarkable formula can be derived from the entropy equation (1.4) utilizing the kinetic formulation or by fundamental solution of linear wave equation. We remark that when $g(\xi)=1$, $\eta(\rho,m)=\rho$; when $g(\xi)=\xi$, $\eta(\rho,m)=m$; and when $g(\xi)=\frac{1}{2}\xi^2$, then $\eta=\dfrac{m^2}{2\rho}+\dfrac{\kappa}{\gamma-1}\rho^\gamma$ is mechanical energy.

As the convexity of entropy function is crucial in the definition of admissible weak solutions, the characterization of convexity of entropy functions is important. In our case, the following lemma provides full details in this direction.

**Lemma 3.1.** *(Lions-Perthame-Tadmor, [24]) Weak entropy $\eta(\rho,m)$ defined in (3.1) is convex with respect to $\rho$ and $m$ if and only if $g(\xi)$ is a convex function.*

We now present the proof for Theorem 1.1.

### *3.1.* Proof of Theorem 1.1:

Choosing $g(\xi)=g_k(\xi)=e^{k\xi^2}$ in (3.1), for positive parameter $k>0$, the corresponding entropy $\eta_k\geq 0$ is clearly convex. By the definition, if $(\rho,m)$ is a $L^\infty$ entropy weak solution, we have the following entropy inequality

$$\eta_{k,t}+q_{k,x}+\kappa\eta_{k,m}m\leq 0, \tag{3.3}$$

in the sense of distributions. Assuming temporarily that $\eta_{k,m}m\geq 0$, we define

$$M(T)=\|u\|_{L^\infty(\mathbf{R}\times[0,T])}+\theta\|\rho^\theta\|_{L^\infty(\mathbf{R}\times[0,T])},$$

which measures the largest possible amplitude of the characteristic speed of the system (1.1) up to $T>0$. It follows, by the $L^\infty$ divergence-measure field theory [5], that for any $a>0$ and $T>0$,

$$\int_{-a}^{a}\eta_k(\rho,m)(x,T)dx\leq\int_{-a-M(T)T}^{a+M(T)T}\eta_k(\rho,m)(x,0)dx, \tag{3.4}$$

In view of positivity of $\eta_k$, one has

$$\Big(\int_{-a}^{a}\eta_k(\rho,m)(x,T)dx\Big)^{\frac{1}{k}}\leq\Big(\int_{-a-M(T)T}^{a+M(T)T}\eta_k(\rho,m)(x,0)dx\Big)^{\frac{1}{k}}. \tag{3.5}$$

Noting that

$$\eta_k=\rho\int_{-1}^{1}e^{k(u+z\rho^\theta)^2}(1-z^2)^\lambda dz,$$

(3.5) implies, for $Q_a(T)=\{(x,t,z):-a\leq x\leq a, t=T, -1\leq z\leq 1\}$, that

$$\begin{aligned}\|\rho^{\frac{1}{k}}e^{(u+z\rho^\theta)^2}(1-z^2)^{\frac{\lambda}{k}}\|_{L^k(Q_a(T))}&\leq\Big(\int_{-a}^{a}\eta_k(\rho,m)(x,T)dx\Big)^{\frac{1}{k}}\\&\leq\Big(\int_{-a-M(T)T}^{a+M(T)T}\eta_k(\rho,m)(x,0)dx\Big)^{\frac{1}{k}}.\end{aligned} \tag{3.6}$$

Letting $k \to +\infty$ and then $a \to +\infty$ in (3.6), we have

$$\begin{aligned}&\max\{\|(u-\rho^\theta)(x,T)\|_{L^\infty}, \|(u+\rho^\theta)(x,T)\|_{L^\infty}\}\\&\le \max\{\|(u-\rho^\theta)(x,0)\|_{L^\infty}, \|(u+\rho^\theta)(x,0)\|_{L^\infty}\},\end{aligned} \tag{3.7}$$

which implies, for any $T>0$, that

$$0 \le \rho(x,T) \le C,\ 0 \le |u(x,T)| \le C. \tag{3.8}$$

where $C$ only depends on the initial data.

It remains to show $\eta_{k,m} m \ge 0$. In fact,

$$\begin{aligned} m\eta_{k,m} &= 2km\int_{-1}^{1} e^{k(u+z\rho^\theta)^2}(u+z\rho^\theta)(1-z^2)^\lambda dz\\ &= 2km\sum_{n=0}^{\infty}\int_{-1}^{1}\frac{k^n}{n!}(u+z\rho^\theta)^{2n+1}(1-z^2)^\lambda dz\\ &= 2\rho u^2\sum_{n=0}^{\infty}\frac{k^{n+1}}{n!}\sum_{j=0}^{2n+1}C_{2n+1}^{j}\int_{-1}^{1}u^{2n-j}(z\rho^\theta)^j(1-z^2)^\lambda dz\\ &= 2\rho u^2\sum_{n=0}^{\infty}\frac{k^{n+1}}{n!}\sum_{j=0}^{n}C_{2n+1}^{2j}\int_{-1}^{1}u^{2(n-j)}(z\rho^\theta)^{2j}(1-z^2)^\lambda dz\\ &\ge 0.\end{aligned} \tag{3.9}$$

This concludes the proof of Theorem 1.1.

## 4. Decay estimates

We now proceed to the decay estimates. To begin, we first prove two important inequalities which give sharp information on the pressure near vacuum, where the theory is most lacking.

**Lemma 4.1.** *If* $0 \le \rho, \bar\rho \le C$, *there are two constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\begin{cases} c_1(\rho^{\gamma-1}+\bar\rho^{\gamma-1})(\rho-\bar\rho)^2 \le \rho^{\gamma+1}-\bar\rho^{\gamma+1}-(\gamma+1)\bar\rho^\gamma(\rho-\bar\rho)\\ \qquad\qquad\qquad\qquad\qquad \le c_2(\rho^{\gamma-1}+\bar\rho^{\gamma-1})(\rho-\bar\rho)^2\\ c_1(\rho^{\gamma-1}+\bar\rho^{\gamma-1})(\rho-\bar\rho)^2 \le (\rho^\gamma-\bar\rho^\gamma)(\rho-\bar\rho)\\ \qquad\qquad\qquad\qquad\qquad \le c_2(\rho^{\gamma-1}+\bar\rho^{\gamma-1})(\rho-\bar\rho)^2.\end{cases} \tag{4.1}$$

**Proof:** By Taylor theorem, we have

$$\begin{aligned}&\rho^{\gamma+1}-\bar\rho^{\gamma+1}-(\gamma+1)\bar\rho^\gamma(\rho-\bar\rho)\\ &\qquad = \gamma(\gamma+1)(\rho-\bar\rho)^2\Big[\int_0^1(1-s)((1-s)\bar\rho+s\rho)^{\gamma-1}\,ds\Big],\\ &(\rho^\gamma-\bar\rho^\gamma)(\rho-\bar\rho)\\ &= (\rho-\bar\rho)^2\gamma\Big[\int_0^1((1-s)\bar\rho+s\rho)^{\gamma-1}\,ds\Big].\end{aligned} \tag{4.2}$$

Noting that for any $0 < s < 1$,

$$\max\{((1-s)\bar{\rho})^{\gamma-1},(s\rho)^{\gamma-1}\} \le ((1-s)\bar{\rho}+s\rho)^{\gamma-1} \le \max\{\bar{\rho}^{\gamma-1},\rho^{\gamma-1}\}, \quad (4.3)$$

(4.1) follows easily from (4.2) and (4.3).

We are now ready to derive the decay estimates. Suppose that $(\rho,m)$ is a weak entropy solution of (1.1) satisfying conditions in Theorem 1.2, then $(\rho,m)$ satisfies

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (\dfrac{m^2}{\rho} + \kappa(\rho^\gamma))_x = -\kappa m. \end{cases} \quad (4.4)$$

Let $\bar{\rho}$ be the Barenblatt's solution of porous medium equation carrying the same total mass $M$ as $\rho$, and $\bar{m} = -(\bar{\rho}^\gamma)_x$, then

$$\begin{cases} w = \rho - \bar{\rho}, \\ z = m - \bar{m}, \end{cases} \quad (4.5)$$

satisfying

$$\begin{cases} w_t + z_x = 0 \\ z_t + (\dfrac{m^2}{\rho})_x + k(\rho^\gamma - \bar{\rho}^\gamma)_x + \kappa z = -\bar{m}_t. \end{cases} \quad (4.6)$$

Setting

$$y = -\int_{-\infty}^{x} w(r,t)dr, \quad (4.7)$$

we have

$$y_x = -w, \;\; z = y_t. \quad (4.8)$$

Thus the equation (4.6) turns into a nonlinear wave equation with source term, degenerate at vacuum:

$$y_{tt} + (\frac{m^2}{\rho})_x + \kappa(\rho^\gamma - \bar{\rho}^\gamma)_x + \kappa y_t = -\bar{m}_t. \quad (4.9)$$

Multiplying $y$ with (4.9), integrating over $[0,t]\times(-\infty,\infty)$, integrating by parts, one has

$$\begin{aligned} &\int_{-\infty}^{+\infty}(y_t y + \frac{\kappa}{2}y^2)\,dx + \int_0^t\int_{-\infty}^{+\infty}\kappa(\rho^\gamma - \bar{\rho}^\gamma)(\rho-\bar{\rho})\,dxd\tau \\ &\le C + \int_0^t\int_{-\infty}^{+\infty}(y_t^2 + \frac{m^2}{\rho}y_x)\,dxd\tau + |\int_0^t\int_{-\infty}^{+\infty}\bar{m}_t y\,dxd\tau|. \end{aligned} \quad (4.10)$$

Since $\|y_x\|_{L^1} = \|\rho-\bar{\rho}\|_{L^1} \le 2M$, we have, from (2.7), the following estimate

$$\begin{aligned} |\int_0^t\int_{-\infty}^{+\infty}\bar{m}_t y\,dxd\tau| &= |\int_0^t\int_{-\infty}^{+\infty}\bar{\rho}_t^\gamma y_x\,dxd\tau| \\ &\le C\int_0^t(1+\tau)^{-1-\frac{\gamma}{\gamma+1}}\,d\tau \le C. \end{aligned} \quad (4.11)$$

We thus obtain our first estimate

**Lemma 4.2.** *Let the conditions of Theorem 1.2 be satisfied, it holds*

$$\begin{aligned}&\int_{-\infty}^{+\infty}(y_t y+\frac{\kappa}{2}y^2)\,dx+\int_0^t\int_{-\infty}^{+\infty}\kappa(\rho^\gamma-\bar\rho^\gamma)(\rho-\bar\rho)\,dxd\tau\\ &\le C+\int_0^t\int_{-\infty}^{+\infty}y_t^2\,dxd\tau+\int_0^t\int_{-\infty}^{+\infty}\frac{m^2}{\rho}y_x\,dxd\tau.\end{aligned}\tag{4.12}$$

This estimate is very rough in its nature. In order to obtain a definite estimate, higher order estimate is required. Due to the lack of regularity, we will need to close the estimate in the next run, which is achieved by entropy inequality.

Choosing $g(\xi)=\frac{1}{2}\xi^2$ in (3.1), the entropy $\eta$ takes the form of mechanical energy

$$\eta_e=\frac{m^2}{2\rho}+\frac{\kappa}{\gamma-1}\rho^\gamma,$$

and $q_e$ the corresponding flux. We read from the entropy inequality that

$$\eta_{et}+q_{ex}+\kappa\frac{m^2}{\rho}\le 0,\tag{4.13}$$

which implies with the help of the divergence measure field theory of [5] that

$$\int_{-\infty}^{+\infty}\eta_e(x,t)\,dx+\kappa\int_0^t\int_{-\infty}^{+\infty}\frac{m^2}{\rho}\,dxd\tau\le C.\tag{4.14}$$

Since $y_t=m-\bar m$, we thus have

$$\begin{aligned}\int_0^t\int_{-\infty}^{+\infty}y_t^2\,dxd\tau&\le 2\int_0^t\int_{-\infty}^{+\infty}\bar m^2\,dxd\tau+2\int_0^t\int_{-\infty}^{+\infty}m^2\,dxd\tau\\ &\le C\int_0^t(1+\tau)^{-\frac{2\gamma+1}{\gamma+1}}\,d\tau+C\int_0^t\int_{-\infty}^{+\infty}\frac{m^2}{\rho}\,dxd\tau\\ &\le C.\end{aligned}\tag{4.15}$$

and

$$\left|\int_0^t\int_{-\infty}^{+\infty}\frac{m^2}{\rho}y_x\,dxd\tau\right|\le C\int_0^t\int_{-\infty}^{+\infty}\frac{m^2}{\rho}\,dxd\tau\le C.\tag{4.16}$$

Substituting (4.14), (4.15) and (4.16) into (4.12), we arrive at the following uniform estimate

**Lemma 4.3.** *Under the conditions of Theorem 1.2, for any $t>0$, it holds that*

$$\begin{aligned}&\int_{-\infty}^{+\infty}(\rho^\gamma+y_t^2+\frac{m^2}{\rho}+y^2)\,dx+\int_0^t\int_{-\infty}^{+\infty}(\frac{m^2}{\rho}+y_t^2)\,dxd\tau\\ &\quad+\int_0^t\int_{-\infty}^{+\infty}(\rho^\gamma-\bar\rho^\gamma)(\rho-\bar\rho)\,dxd\tau\le C.\end{aligned}\tag{4.17}$$

**Remark 4.1.** It holds for the Barenblatt solution that

$$\int_0^\infty \int_{-\infty}^\infty \bar{\rho}^{\gamma+1} dxdt = \infty. \tag{4.18}$$

On the other hand, one has

$$(\rho^\gamma - \bar{\rho}^\gamma)(\rho - \bar{\rho}) \geq |\rho - \bar{\rho}|^{\gamma+1}.$$

(4.17) thus hints $\rho$ tends to the Barenblatt solution time asymptotically. In fact, a further study carried in [21] gives a rough decay rates on $\|\rho - \bar{\rho}\|_{L^\gamma}$. We remark that $\eta_e$ measures $\rho^\gamma$ while the last term on the left hand side of (4.17) is in the form of $L^{\gamma+1}$. The mis-match of the exponent leads to essential difficulty for better decay rates.

For sharper decay rate, we shall use the entropy inequality again which measures $L^{\gamma+1}$ norm in density. Choosing $g(\xi) = |\xi|^{\frac{2\gamma}{\gamma-1}}$ in (3.1), the entropy reads as

$$\tilde{\eta} = \rho \int_{-1}^1 |u + z\rho^\theta|^{\frac{2\gamma}{\gamma-1}} (1-z^2)^\lambda dz. \tag{4.19}$$

We shall see below that such an entropy function measures $L^{\gamma+1}$ norm of $\rho$, and thus matches the double integral term in (4.17).

By Taylor theorem, we have

$$\begin{aligned}
\tilde{\eta} &= \rho \int_{-1}^1 g(u + z\rho^\theta)(1-z^2)^\lambda dz \\
&= \rho \int_{-1}^1 [g(z\rho^\theta) + g'(z\rho^\theta)u + \frac{1}{2} g''(z\rho^\theta)u^2](1-z^2)^\lambda dz \\
&\quad + \rho \int_{-1}^1 [g(u + z\rho^\theta) - g(z\rho^\theta) - g'(z\rho^\theta)u - \frac{1}{2} g''(z\rho^\theta)u^2](1-z^2)^\lambda dz \\
&=: C_1 \rho^{\gamma+1} + C_2 m^2 + A(\rho, m),
\end{aligned} \tag{4.20}$$

where

$$\begin{aligned}
A(\rho, m) &= \rho \int_{-1}^1 [g(u + z\rho^\theta) - g(z\rho^\theta) - g'(z\rho^\theta)u - \frac{1}{2} g''(z\rho^\theta)u^2](1-z^2)^\lambda dz \\
&= \rho u^3 \int_{-1}^1 \int_0^1 \frac{(1-s)^2}{2} g^{(3)}(su + z\rho^\theta)(1-z^2)^\lambda dsdz,
\end{aligned} \tag{4.21}$$

$$\begin{cases}
C_1 = \int_{-1}^1 |z|^{\frac{2\gamma}{\gamma-1}} (1-z^2)^\lambda dz = \frac{1}{2} B(\frac{\gamma+1}{2(\gamma-1)}, \frac{\gamma+1}{2(\gamma-1)}), \\
C_2 = \frac{\gamma(\gamma+1)}{(\gamma-1)^2} \int_{-1}^1 |z|^{\frac{2\gamma}{\gamma-1}} (1-z^2)^\lambda dz = 2 \frac{\gamma(\gamma+1)}{(\gamma-1)^2} C_1
\end{cases} \tag{4.22}$$

and $B(p,q)$ is Beta function defined by

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx.$$

Here we have used the following fact

$$\int_{-1}^{1} g'(z\rho^\theta)u(1-z^2)^\lambda dz = \frac{2\gamma}{\gamma-1}\rho^\theta u \int_{-1}^{1} |z\rho^\theta|^{\frac{2}{\gamma-1}} z(1-z^2)^\lambda dz = 0. \qquad (4.23)$$

Further advantages of this entropy $\tilde{\eta}$ are summarized in the following lemma.

**Lemma 4.4.** *For $A(\rho,m)$ defined in (4.21), it holds that*

- *1)* $|A| \le C\rho|u|^3(|u|^{\frac{3-\gamma}{\gamma-1}} + \rho^{1-\theta})$,
- *2)* $A(\rho,m) \ge 0$,
- *3)* $A_m m \ge 0$.

**Proof:** From the formula (4.21) of $A(\rho,m)$, it is obvious that

$$|A(\rho,m)| \le C\rho|u|^3|g^{(3)}(|u|+\rho^\theta)| \le C\rho|u|^3(|u|^{\frac{3-\gamma}{\gamma-1}} + \rho^{1-\theta}).$$

We now prove 2) and 3). In terms of (4.21) and the fact

$$\int_{-1}^{1} g^{(3)}(z\rho^\theta)(1-z^2)^\lambda \, dz = 0,$$

we get

$$\begin{aligned} A &= \rho u^3 \int_{-1}^{1}\int_0^1 \frac{(1-s)^2}{2}[g^{(3)}(su+z\rho^\theta) - g^{(3)}(z\rho^\theta)](1-z^2)^\lambda ds dz \\ &= \rho u^4 \int_{-1}^{1}\int_0^1\int_0^1 \frac{(1-s_1)^2 s_2}{2} g^{(4)}(s_1 s_2 u + z\rho^\theta)(1-z^2)^\lambda ds_1 ds_2 dz \ge 0 \end{aligned} \qquad (4.24)$$

and

$$A_m m = 3A + \rho u^4 \int_{-1}^{1}\int_0^1 \frac{s(1-s)^2}{2} g^{(4)}(su+z\rho^\theta)(1-z^2)^\lambda ds dz \ge 0 \qquad (4.25)$$

where we have used $g^{(4)} \ge 0$. Thus Lemma 4.4 is proved.

Let $\tilde{q}$ be the flux corresponding to entropy $\tilde{\eta}$. Define

$$\eta_* = \tilde{\eta} - C_1\bar{\rho}^{\gamma+1} - C_1(\gamma+1)\bar{\rho}^\gamma(\rho-\bar{\rho}) \qquad (4.26)$$

where $\bar{\rho}$ is the Barenblatt solution defined in (2.2). The entropy inequality implies

$$\eta_{*t} + (C_1\bar{\rho}^{\gamma+1} + C_1(\gamma+1)\bar{\rho}^\gamma(\rho-\bar{\rho}))_t + \tilde{q}_x + 2\kappa C_2 m^2 + \kappa A_m m \le 0. \qquad (4.27)$$

Note that

$$C_1(\gamma+1) = 2\kappa C_2 \qquad (4.28)$$

due to (4.22), and

$$(\bar{\rho}^{\gamma+1})_t = (\gamma+1)\bar{\rho}^{\gamma}\bar{\rho}_t = -(\gamma+1)\bar{m}^2 + (\cdots)_x, \tag{4.29}$$

where $(\cdots)_x$ denotes terms vanish after integrating over $\mathbf{R}$. We thus update (4.27) by

$$\begin{aligned} \eta_{*t} + (2\kappa C_2\bar{\rho}^{\gamma}(\rho-\bar{\rho}))_t + 2\kappa C_2(m-\bar{m})^2 + \\ 4\kappa C_2\bar{m}(m-\bar{m}) + \kappa A_m m + (\cdots)_x \le 0. \end{aligned} \tag{4.30}$$

Since $\rho - \bar{\rho} = -y_x$ and

$$\bar{m}(m-\bar{m}) = -(\bar{\rho}^{\gamma})_x y_t = -(\bar{\rho}_x^{\gamma} y)_t + (\bar{\rho}^{\gamma})_t(\rho-\bar{\rho}) + (\cdots)_x, \tag{4.31}$$

(4.30) is reduced into

$$\begin{aligned} \eta_{*t} - 2\kappa C_2(\bar{\rho}_x^{\gamma} y)_t + 2\kappa C_2(m-\bar{m})^2 + 4\kappa C_2(\bar{\rho}^{\gamma})_t(\rho-\bar{\rho}) \\ + \kappa A_m m + (\cdots)_x \le 0. \end{aligned} \tag{4.32}$$

For any small positive constant $\varepsilon > 0$, we define

$$\mu(\varepsilon) = 1 - \frac{1}{2(\gamma+1)} - \varepsilon.$$

Multiplying (4.32) by $(1+t)^{\mu(\varepsilon)}$, integrating the result on $\mathbf{R}\times[0,t]$ and using Lemma 4.4, we find

$$\begin{aligned} &(1+t)^{\mu(\varepsilon)}\int_{-\infty}^{+\infty}\eta_*\,dx + \int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\mu(\varepsilon)}[2\kappa C_2(m-\bar{m})^2 + \kappa A_m m]\,dxd\tau \\ &\le C + C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\mu(\varepsilon)-1}\eta_* dxd\tau + C(1+t)^{\mu(\varepsilon)}\int_{-\infty}^{+\infty}|\bar{\rho}_x^{\gamma}||y|\,dx \\ &+ C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\mu(\varepsilon)-1}|\bar{\rho}_x^{\gamma}||y|\,dxd\tau + C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\mu(\varepsilon)}|\bar{\rho}_t^{\gamma}||\rho-\bar{\rho}|\,dxd\tau \\ &=: C + \sum_{i=1}^4 I_i. \end{aligned} \tag{4.33}$$

From (1.5), (4.20), (4.26), (4.1) and Lemma 4.4, we have

$$\eta_* \le C(\rho^{\gamma}-\bar{\rho}^{\gamma})(\rho-\bar{\rho}) + C\frac{m^2}{\rho}, \tag{4.34}$$

which, together with Lemma 4.3, implies

$$\begin{aligned} I_1 &\le C\int_0^t\int_{-\infty}^{+\infty}\eta_*\,dxd\tau \\ &\le C\int_0^t\int_{-\infty}^{+\infty}((\rho^{\gamma}-\bar{\rho}^{\gamma})(\rho-\bar{\rho}) + \frac{m^2}{\rho})\,dxd\tau \le C. \end{aligned} \tag{4.35}$$

By Lemma 4.3 again, we have

$$
\begin{aligned}
I_2 &\le C\int_{-\infty}^{+\infty} y^2 dx + C(1+t)^{2\mu(\varepsilon)}\int_{-\infty}^{+\infty}|\bar\rho_x^\gamma|^2 dx \\
&\le C + C(1+t)^{2\mu(\varepsilon)-2+\frac{1}{\gamma+1}} \\
&\le C + C(1+t)^{-2\varepsilon} \le C.
\end{aligned}
\tag{4.36}
$$

Similarly, we have

$$
\begin{aligned}
I_3 &\le C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\mu(\varepsilon)-1}|\bar\rho_x^\gamma||y|\,dxd\tau \\
&\le \int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{-1-\varepsilon}y^2\,dxd\tau + C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{2\mu(\varepsilon)-1+\varepsilon}|\bar\rho_x^\gamma|^2\,dxd\tau \\
&\le \int_0^t(1+\tau)^{-1-\varepsilon}\,d\tau + C\int_0^t(1+\tau)^{2\mu(\varepsilon)-3+\varepsilon+\frac{1}{\gamma+1}}\,d\tau \\
&\le C + C\int_0^t(1+\tau)^{-1-\varepsilon}\,d\tau \\
&\le C.
\end{aligned}
$$

It remains to deal with the term $I_4$. From (2.2) and (4.1), we obtain, when $1<\gamma<3$,

$$
\begin{aligned}
I_4 &\le C\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{\mu(\varepsilon)}|\bar\rho_t^\gamma||\rho-\bar\rho|\,dxd\tau \\
&\le C\int_0^t\int_{-\infty}^{+\infty}\bar\rho^{\gamma-1}(\rho-\bar\rho)^2\,dxd\tau \\
&\quad + C\textstyle\int_0^t\int_{-\infty}^{+\infty}(1+\tau)^{2\mu(\varepsilon)}\bar\rho^{1-\gamma}|\bar\rho_t^\gamma|^2\,dxd\tau \\
&\le C + C\int_0^t(1+\tau)^{2\mu(\varepsilon)-2-\frac{\gamma}{\gamma+1}}d\tau \\
&\le C + C\int_0^t(1+\tau)^{-1-2\varepsilon}\,d\tau \le C.
\end{aligned}
\tag{4.37}
$$

Thus, in view of (4.33), (4.35), (4.36) and (4.37), we get

$$
\int_{-\infty}^{+\infty}\eta_*\,dx \le C(1+t)^{-\mu(\varepsilon)}. \tag{4.38}
$$

Therefore we can prove the following lemma,

**Lemma 4.5.** *Under the conditions of Theorem 1.2, it holds for any $t>0$ that*

$$
\begin{aligned}
&\|(m-\bar m)(\cdot,t)\|_{L^2}^2 + \|(\rho-\bar\rho)(\cdot,t)\|_{L^{\gamma+1}}^{\gamma+1} \\
&\quad + \int_{-\infty}^{+\infty}(\rho^{\gamma-1}+\bar\rho^{\gamma-1})(\rho-\bar\rho)^2\,dx \le C(1+t)^{\frac{1}{2(\gamma+1)}+\varepsilon-1}, \\
&\int_0^t(1+\tau)^{1-\frac{1}{2(\gamma+1)}-\varepsilon}\|(m-\bar m)(\cdot,\tau)\|_{L^2}^2\,d\tau \le C,
\end{aligned}
\tag{4.39}
$$

*for any positive constant $\varepsilon$.*

**Proof.** From Lemma 4.1 and the following fact

$$(\rho^{\gamma-1}+\bar{\rho}^{\gamma-1})(\rho-\bar{\rho})^2 \geq C(\rho^\gamma-\bar{\rho}^\gamma)(\rho-\bar{\rho}) \geq C|\rho-\bar{\rho}|^{\gamma+1},$$

the estimates on $\rho$ is directly from (4.39). It remains to show the estimate on $m$. For this purpose, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}(m-\bar{m})^2\,dx &\leq C\int_{-\infty}^{+\infty}(m^2+\bar{m}^2)\,dx\\ &\leq C\int_{-\infty}^{+\infty}\eta_*\,dx+\int_{-\infty}^{+\infty}(\bar{\rho}^\gamma)_x^2\,dx\\ &\leq C(1+t)^{\frac{1}{2(\gamma+1)}+\varepsilon-1}+C(1+t)^{-\frac{2\gamma+1}{\gamma+1}}\\ &\leq C(1+t)^{\frac{1}{2(\gamma+1)}+\varepsilon-1},\end{aligned}$$

where we have used the fact that

$$-\frac{2\gamma+1}{\gamma+1}<\frac{1}{2(\gamma+1)}-1=-\frac{2\gamma+1}{2(\gamma+1)}.$$

The proof of Lemma 4.5 is complete.

Next, we shall use Lemma 4.5 to further obtain an $L^1$ convergence rate on density, which is based on the following key observation.

**Lemma 4.6.** *If $\rho\geq 0$ and $\bar{\rho}\geq 0$ have the same total mass $M$, then for any $t>0$,*

$$\int_{-\infty}^{+\infty}|\rho-\bar{\rho}|(x,t)dx \leq 2\int_{\bar{\rho}>0}|\rho-\bar{\rho}|(x,t)dx. \tag{4.40}$$

**Proof:** Because

$$\int_{-\infty}^{+\infty}\rho(x,t)\,dx=\int_{\bar{\rho}>0}\rho dx+\int_{\bar{\rho}=0}\rho dx=\int_{-\infty}^{+\infty}\bar{\rho}\,dx=\int_{\bar{\rho}>0}\bar{\rho}dx, \tag{4.41}$$

we have

$$\int_{\bar{\rho}=0}\rho\,dx=\int_{\bar{\rho}>0}(\bar{\rho}-\rho)\,dx\leq\int_{\bar{\rho}>0}|\bar{\rho}-\rho|\,dx \tag{4.42}$$

and

$$\int_{-\infty}^{+\infty}|\rho-\bar{\rho}|\,dx=\int_{\bar{\rho}>0}|\rho-\bar{\rho}|\,dx+\int_{\bar{\rho}=0}\rho\,dx\leq 2\int_{\bar{\rho}>0}|\rho-\bar{\rho}|\,dx. \tag{4.43}$$

Therefore Lemma 4.6 is complete.

**Remark 4.2.** Lemma 4.6 discloses the fact that $|\rho-\bar{\rho}|$ on the support of the Barenblatt solution $\bar{\rho}$ plays a leading role in the $L^1$ estimate. It is noted that the support of $\bar{\rho}$ is a bounded domain for any fixed $t>0$, thus it is possible to obtain an $L^1$ convergence rate from $L^p$ convergence rate.

The decay rates of the $L^1$ distance between $\rho$ and $\bar{\rho}$ is given in the next lemma.

**Lemma 4.7.** *Assume the conditions in Theorem 1.2 are satisfied, then*

$$\|\rho-\bar\rho\|_{L^1}\le C(1+t)^{-\frac{1}{4(\gamma+1)}+\varepsilon},\ \forall t>0,$$

*for any* $\varepsilon>0$.

**Proof:** To take care of the singularity of $\bar\rho$, we divide the support of $\bar\rho$ into two parts:

$$\begin{aligned}
&\Omega_0=(-\sqrt{\frac{A}{B}}(1+t)^{\frac{1}{\gamma+1}}+(1+t)^{-\beta},\sqrt{\frac{A}{B}}(1+t)^{\frac{1}{\gamma+1}}-(1+t)^{-\beta}),\\
&\Omega_1=(-\sqrt{\frac{A}{B}}(1+t)^{\frac{1}{\gamma+1}},-\sqrt{\frac{A}{B}}(1+t)^{\frac{1}{\gamma+1}}+(1+t)^{-\beta})\\
&\cup(\sqrt{\frac{A}{B}}(1+t)^{\frac{1}{\gamma+1}}-(1+t)^{-\beta},\sqrt{\frac{A}{B}}(1+t)^{\frac{1}{\gamma+1}}),
\end{aligned}\tag{4.44}$$

where $\beta>1$ is a constant. Then $\{\bar\rho>0\}=\Omega_0\cup\Omega_1$.

With the help of Lemma 4.5 and Lemma 4.6, we now compute

$$\begin{aligned}
&\int_{\bar\rho>0}|\rho-\bar\rho|dx\\
&=\int_{\Omega_0}|\rho-\bar\rho|dx+\int_{\Omega_1}|\rho-\bar\rho|dx\\
&\le(\int_{\Omega_0}\bar\rho^{\gamma-1}|\rho-\bar\rho|^2dx)^{\frac12}(\int_{\Omega_0}\bar\rho^{1-\gamma}dx)^{\frac12}+\int_{\Omega_1}|\rho-\bar\rho|dx\\
&\le C(1+t)^{-\frac12+\frac{1}{4(\gamma+1)}+\frac{\varepsilon}{2}}(1+t)^{\frac{\gamma-1}{2(\gamma+1)}}(\int_{\Omega_0}(A-\frac{Bx^2}{(1+t)^{\frac{2}{\gamma+1}}})^{-1}dx)^{\frac12}+C(1+t)^{-\beta}\\
&\le C(1+t)^{-\frac{1}{4(\gamma+1)}+\frac{\varepsilon}{2}}\sqrt{\ln(1+t)}\le C(1+t)^{-\frac{1}{4(\gamma+1)}+\varepsilon}.
\end{aligned}\tag{4.45}$$

Here we have used the fact that

$$\begin{aligned}
&|\int_{\Omega_0}(A-\frac{Bx^2}{(1+t)^{\frac{2}{\gamma+1}}})^{-1}dx|\\
&=B^2(1+t)^{\frac{1}{\gamma+1}}\int_{\xi=-\sqrt{\frac{A}{B}}+(1+t)^{-\beta-\frac{1}{\gamma+1}}}^{\xi=\sqrt{\frac{A}{B}}-(1+t)^{-\beta-\frac{1}{\gamma+1}}}(\sqrt{\frac{A}{B}}-\xi)^{-1}(\sqrt{\frac{A}{B}}+\xi)^{-1}\,d\xi\\
&\le C(1+t)^{\frac{1}{\gamma+1}}\ln(1+t).
\end{aligned}$$

This completes the proof of Lemma 4.7.

Theorem 1.2 follows from Lemmas 4.5 and 4.7.

## 5. Remarks and discussions

This section is devoted to make some further remarks on two issues. The first is about an alternative asymptotic profiles of (1.1) constructed by T. Liu. The other one is about the optimality of the rates in our Theorem 1.2.

### *5.1. Liu's solution*

In [26], Liu constructed an interesting solution to (1.1) mimic the Barenblatt's solution to (1.2). In this section, we shall discuss Liu's solution in its one dimensional version and the relation to our studies in this section.

Define $\phi = p'(\rho) = k\gamma\rho^{\gamma-1}$ be the square of the sound speed, one can rewrite (1.1) in terms of $u$ and $\phi$:

$$\begin{cases} \phi_t + u\phi_x + (\gamma-1)\phi u_x = 0 \\ u_t + uu_x + \frac{1}{\gamma-1}\phi_x = -\alpha u. \end{cases} \tag{5.1}$$

Consider solutions with finite mass such that

$$\rho(x,t) \equiv 0, \text{ for } |x| \geq (\frac{e(t)}{b(t)})^{\frac{1}{2}},$$

and set

$$\phi(x,t) = [e(t) - b(t)x^2]_+, \; u(x,t) = a(t)x, \tag{5.2}$$

for some non-negative smooth functions $a(t)$, $b(t)$ and $e(t)$ to be determined. One thus obtains the following ordinary differential equations for the functions $a(t)$, $b(t)$ and $e(t)$:

$$\begin{cases} e' + (\gamma-1)ae = 0 \\ b' + (\gamma+1)ab = 0 \\ a' + a^2 + \alpha a - \frac{2}{\gamma-1}b = 0. \end{cases} \tag{5.3}$$

The first two equations imply that

$$\left[\ln(\frac{e^{\gamma+1}}{b^{\gamma-1}})\right]' = 0, \tag{5.4}$$

which induces that

$$e^{\gamma+1} = e_1 b^{\gamma-1}, \tag{5.5}$$

for some constant $e_1 > 0$. Therefore, one only needs to solve the equations for $a$ and $b$ in (5.3). The existence of global solution to (5.3) thus could be done by the phase-plane diagram analysis on the region

$$a \geq 0, \; b \geq 0,$$

where $(0,0)$ is the only stable equilibrium. Furthermore, $b' < 0$ inside the region while $a'$ changes sign across the curve

$$\Gamma_1 : b = \frac{\gamma-1}{2}(a^2 + \alpha a).$$

Indeed, $a' > 0$ above $\Gamma_1$, and $a' < 0$ below $\Gamma_1$. A further conclusion from this analysis leads to $(a,b)(t) \to 0$ as $t \to \infty$.

The above procedure can be performed for the porous medium equation to obtain the Barenblatt's solution with the same ansatz as (5.2):

$$\bar{\phi} = [\bar{e}(t) - \bar{b}(t)x^2]_+, \ \bar{u} = \bar{a}(t)x. \tag{5.6}$$

Instead of (5.3), for porous medium equation, one has

$$\begin{cases} \bar{e}' + (\gamma-1)\bar{a}\bar{e} = 0 \\ \bar{b}' + (\gamma+1)\bar{a}\bar{b} = 0 \\ \alpha\bar{a} - \frac{2}{\gamma-1}\bar{b} = 0, \end{cases} \tag{5.7}$$

which is solved explicitly as

$$\begin{cases} \bar{a}(t) = \frac{1}{\gamma+1}t^{-1} \\ \bar{b}(t) = \frac{\alpha(\gamma-1)}{2(\gamma+1)}t^{-1} \\ \bar{e}(t) = e_0 t^{-\frac{\gamma-1}{\gamma+1}}, \end{cases} \tag{5.8}$$

if the initial density is chosen as a pointed mass located at the origin. From this solution, one has

$$\bar{\phi} = t^{-\frac{\gamma-1}{\gamma+1}}[e_0 - \frac{\alpha(\gamma-1)}{\gamma+1}\xi^2]_+, \ \bar{\rho} = (\frac{\bar{\phi}}{k\gamma})^{\frac{1}{\gamma-1}} = t^{-\frac{1}{\gamma+1}}[A - B\xi^2]_+^{\frac{1}{\gamma-1}}, \tag{5.9}$$

where $e_0$ is chosen as $k\gamma A$, while $A$, $B$ and $\xi$ were defined in Section 2.

When the initial data for $a(t)$, $b(t)$ and $e(t)$ are chosen such that the initial density is also a pointed mass at the origin with the same total mass as the Barenblatt's defined through $(\bar{a}, \bar{b}, \bar{e})$, these two solutions share the very similar behavior. The main result of [26] is that $(a, b, e)$ approaches to $(\bar{a}, \bar{b}, \bar{e})$ time asymptotically with very fast decay rates, which is restated in the following theorem.

**Theorem 5.1.** *(T.P. Liu [26]) If the solution $(\rho, u)(x,t)$ of (1) constructed in terms of a, b and e as solution of (5.3) carries the same total mass M as $(\bar{\rho}, \bar{u})$ given by $\bar{a}$, $\bar{b}$ and $\bar{e}$, then as $t \to +\infty$, it holds*

$$\frac{b(t)}{a(t)} \to \frac{\bar{b}}{\bar{a}} = \frac{\alpha(\gamma-1)}{2}, \tag{5.10}$$

*and*

$$(a(t), b(t), e(t)) = (\bar{a}(t), \bar{b}(t), \bar{e}(t))(1 + O(1)\frac{\ln t}{t}), \tag{5.11}$$

*where $O(1)$ is independent of $t \geq 1$, but varies with the trajectories of (5.3).*

**Remark 5.1.** Theorem 5.1 is sharper than the original statement in [26], but (5.11) should be the true statement as it is clearly stated in the proof of [26]. On the other hand, there seems some misprint in the original statement, as both $\bar{a}(t)$ and $\bar{b}(t)$ decay faster than the remainder term $O(1)\frac{\ln t}{t}$ there.

The detailed information of Liu's solution, denoted by $(\rho_L, u_L)(x,t)$ given in Theorem 5.1 indicates that

$$|(\rho_L-\bar{\rho})(x,t)| \le C[\frac{\ln(t+1)}{t+1}]^{\frac{1}{\gamma-1}}\bar{\rho}(x,t), \tag{5.12}$$

which implies that

$$\|(\rho_L-\bar{\rho})(\cdot,t)\|_{L^1} \le C[\frac{\ln(t+1)}{t+1}]^{\frac{1}{\gamma-1}}. \tag{5.13}$$

Now, one easily concludes from (5.13) and Theorem 1.2 that

$$\|(\rho-\rho_L)(\cdot,t)\|_{L^1} \le C(1+t)^{-\frac{1}{4(\gamma+1)}+\varepsilon}, \text{ for } \forall\varepsilon>0. \tag{5.14}$$

*5.2. Optimality of Decay rates*

It is not clear whether the decay rates we obtained in Theorem 1.2 is optimal or not. At first glance of the rates in (5.12) and (5.13), one may feel that the rates in Theorem 1.2 are too slow. Indeed, since Liu's solution is a particular solution with the same initial data as Barenblatt's solution, the decay rates obtained in (5.12)-(5.13) are much faster than those stated in Theorem 1.2.

However, Theorem 1.2 is general in its nature. First of all, the results in Theorem 1.2 are valid for any $L^\infty$ entropy weak solutions without any regularity, while Liu's solution is Lipschitz almost everywhere and is Hölder at the vacuum boundary. Other reasons include the generic condition of the initial mass distribution in Theorem 1.2, without specification of the center of the mass. We note that a shift of the center of the mass could result big error comparing to the specific choice made in Liu's solution which matches the Barenblatt's solution in a perfect way. For instance, one could easily estimate the $L^1$ decay between Barenblatt's solution and a shifted one as follows.

$$\begin{aligned} \|(\bar{\rho}(x,t)-\bar{\rho}(x+d,t)\|_{L^1} &= O(1)(1+t)^{-\frac{1}{\gamma+1}}\int_0^{\sqrt{\frac{A}{B}}}(\sqrt{\frac{A}{B}}-\xi)^{\frac{1}{\gamma-1}-1}\,d\xi \\ &= O(1)(1+t)^{-\frac{1}{\gamma+1}}. \end{aligned} \tag{5.15}$$

This partially explains why the decay rate of Theorem 1.2 is much slower than (5.11).

We further remark that one should not expect the rate in (5.15) for general case in our problem and for porous medium equation. In fact, Carrillo and Toscani [4] proved that the $L^1$ convergence rate from solutions of porous medium equation with finite toal mass and finite second moment to Barenblatt solution is about $(1+t)^{-\frac{1}{3\gamma-1}}$ using the relation between porous medium equation and the Fokker-Planck equation. This idea is later adopted to the recent work of Ogawa [37] for Keller-Segel system. This rate is much closer to our rates in Theorem 1.2. The difference between these two rates could be explained by many reasons. A distinct feature here is the lack of the regularity in our $L^\infty$ solutions. It is also remarkable that we did not ask the condition on the boundness of second moment. Furthermore, [4] is on the difference between solutions of porous medium equation, while we compare the solutions of compressible Euler equations with damping, which is hyperboic, with the Barenblatt's solution. One thus expects slower decay rates in our case than in [4]. Finally, we note the constraints in our proof for better decay rates are from the terms of $I_i$ $(i=1,2,3,4)$ in (4.33)–(4.38). It is clear that the current rates are very hard to improve with current approach.

On the other hand, we did not find an argument on the optimality of our rates in Theorem 1.2. Therefore, the optimal decay rate of $\|\rho-\bar{\rho}\|_{L^1}$ remains as an interesting open problem.

*Acknowledgements.* F. Huang's research is supported in part by NSFC Grant No. 10825102 for distinguished youth scholar, NSFC-NSAF Grant No. 10676037 and 973 project of China, Grant No. 2006CB805902. R. Pan's research is supported by National Science Foundation under grant DMS-0807406. Zhen Wang's research is supported in part by NSFC Grant No. 10871199.

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