MATH 2551, Spring 2018
Practice Final : Solutions

Problem 1. This problem is about the function

$$f(x, y, z) = 3zy + 4xcos(z).$$

(a) Find the rate of change of the function f at $(1, 1, 0)$ in the direction from this point to the origin.

Solution: The direction vector is $\mathbf{v} = -\mathbf{i} - \mathbf{j}$. Normalize it one obtains:$\mathbf{u} = -\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Compute the gradient of f at $(1, 1, 0)$, we have

$$\nabla f(1, 1, 0) = 4\mathbf{i} + 3\mathbf{k}.$$

Thus: $f'_{\mathbf{u}}(1, 1, 0) = \nabla f(1, 1, 0) \bullet \mathbf{u} = -\frac{4}{\sqrt{2}}$.

(b) Give an approximate value of $f(0.9, 1.2, 0.11)$

Solution: To approximate $f(0.9, 1.2, 0.11)$, we use differentials. We know that $f(1, 1, 0) = 4$, and $\Delta x = -0.1$, $\Delta y = 0.2$, $\Delta z = 0.11$. Thus,

$$f(0.9, 1.2, 0.11) \approx f(1, 1, 0) + df = 4 + 4(-0.1) + 0(0.2) + 3(0.11) = 3.93.$$

(c) The equation $f(x, y, z) = 4$ implicitly defines z as a function of (x, y), if we agree that $z = 0$ if $(x, y) = (1, 1)$. Find the numnerical values of the derivatives:

$\frac{\partial z}{\partial x}(1,1)$ and $\frac{\partial z}{\partial y}(1,1)$.

Solution: By the implicit differentiation, we have

$$\frac{\partial z}{\partial x}(1,1) = -\frac{\partial f/\partial x(1,1,0)}{\partial f \partial z(1,1,0)} = -\frac{4}{3}$$

$$\frac{\partial z}{\partial y}(1,1) = -\frac{\partial f/\partial y(1,1,0)}{\partial f \partial z(1,1,0)} = -\frac{0}{3} = 0.$$

(d) Suppose $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a parametric curve such that $\mathbf{r}(0) = (1,1,0)$ and $\mathbf{r}'(0) = (3,2,1)$. Find the value of

$$\frac{d}{dt} f(\mathbf{r}(t))|_{t=0}.$$

Solution: By chain rule,

$$\frac{d}{dt} f(\mathbf{r}(t))|_{t=0} = \nabla f(\mathbf{r}(0) \bullet \mathbf{r}'(0) = (4\mathbf{i} + 3\mathbf{k}) \bullet (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 15.$$

Problem 2. Consider the planar vector field

$\mathbf{F}(x, y) = (y-2x^2)\mathbf{i}+4\mathbf{j}$, and $\mathbf{G}(x, y) = (1+xy)e^{xy}\mathbf{i}+x^2e^{xy}\mathbf{j}$, and the curve C from point $A(-2, 0)$ to $B(1, 3)$ that goes along the parabola $y = 4 - x^2$.

(a) (4 pt) Is $\mathbf{F}$ a gradient field? If yes, find a function whose gradient is $\mathbf{F}$.

Solution: Set $p = y - 2x^2$, $q = 4$, we see that

$$\frac{\partial p}{\partial y} = 1 \neq \frac{\partial q}{\partial x} = 0.$$

So, $\mathbf{F}$ is not a gradient field.

(b) (6 pt) Is $\mathbf{G}$ a gradient field? If yes, find a function whose gradient is $\mathbf{G}$.

Solution: Set $P = (1 + xy)e^{xy}$, $Q = x^2e^{xy}$, we compute

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2xe^{xy} + x^2ye^{xy}.$$

So, $\mathbf{G} = \nabla g$ for some g.

We now look for $g(x, y)$. To this purpose, we know from $\frac{\partial g}{\partial y} = Q$ that

$$g(x, y) = xe^{xy} + h(x).$$

However, $\frac{\partial g}{\partial x} = P = (1+xy)e^{xy} = (1+xy)e^{xy}+h'(x)$. This implies $h'(x) = 0$. Thus

$$g(x, y) = xe^{xy} + C$$

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(c) Compute the work done by the field $\mathbf{F}$ along the curve C.

Solution: C can be parametrized by

$$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}, \ -2 \le t \le 1.$$

The work done by $\mathbf{F}$ is

$$\begin{aligned} \textstyle\int_C \mathbf{F} \cdot d\mathbf{r} &= \textstyle\int_C (y - 2x^2)dx + 4dy \\ &= \textstyle\int_{-2}^{1} [(4 - t^2 - 2t^2) + 4(-2t)]dt \\ &= 15. \end{aligned}$$

(d) Compute the work done by the field $\mathbf{G}$ along the curve C.

Solution: Since $\mathbf{G} = \nabla g$, by the fundamental theorem of line integrals, the work done by the $\mathbf{G}$ is

$$\int_C \mathbf{G} \cdot d\mathbf{r} = g(1,3) - g(-2,0) = e^3 + 2.$$

Problem 3. Evaluate $I = \int_{C_R} dx + x^2 y dy$, where C_R is the triangle with vertices $(0,0)$, $(0,R)$, $(R,0)$ oriented counterclockwise.

Solution: A convenient way is to apply Green's Theorem. Set $P = 1$, $Q = x^2 y$, we have

$$\begin{aligned}\oint_{C_R} dx + x^2 y dy &= \int\int_D 2xydxdy = \int_0^R \int_0^{R-y} 2xy\ dxdy \\ &= \int_0^R y(R-y)^2\ dy = \frac{1}{2}R^4 - \frac{2}{3}R^4 + \frac{1}{4}R^4 \\ &= \frac{R^4}{12}.\end{aligned}$$

Problem 4 Let S be the portion of the surface $x = 5 - y^2 - z^2$ in the half space $x \geq 1$, oriented so that the normal vector at $(5,0,0)$ is equal to $\mathbf{i}$. Let $\mathbf{F}(x,y,z) = -\mathbf{i} + \mathbf{j}$ (a constant vector field).

(a) Set up and evaluate the flux of $\mathbf{F}$ across S.

Solution: Step 1: We first paramatrize the surface S by $\mathbf{r}(y,z) = (5 - y^2 - z^2)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $(y,z) \in D$. Here D is the disc

$$y^2 + z^2 \le 4.$$

Step 2: We now compute the fundamental vector product $\mathbf{N(y,z)}$.

$$\mathbf{r}'_y = -2y\mathbf{i} + \mathbf{j},$$

$$\mathbf{r}'_z = -2z\mathbf{i} + \mathbf{k},$$

$$\mathbf{N} = \mathbf{r}'_y \times \mathbf{r}'_z = \mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

We confirm that $\mathbf{N}(0,0) = \mathbf{i}$. Set $\mathbf{n}$ be unit vector normalized from $\mathbf{N}$.

Step 3: We now compute the flux of $\mathbf{F}$ across S:

$$\begin{aligned} the\ flux &= \int\int_S \mathbf{F} \cdot \mathbf{n}\ d\sigma \\ &= \int\int_D \mathbf{F} \cdot \mathbf{N}\ dydz \\ &= \int\int_D (-1 + 2y)\ dydz \\ &= \int_0^{2\pi}\int_0^2 (-1 + 2rcos\theta)\ rdrd\theta \\ &= \int_0^{2\pi} (-2 + \frac{16}{3}cos(\theta))\ d\theta \\ &= -4\pi. \end{aligned}$$

(b) Verify that $\mathbf{F} = \nabla \times \mathbf{G}$, where $\mathbf{G} = z\mathbf{j} - x\mathbf{k}$.

Solution: Obivious, omitted.

(c) Give an alternative calculation of the surface integral of part (a) by applying Stokes' theorem.

Solution: The bounding curve of C is $y^2 + z^2 = 4$ oriented in the counterclockwise direction coresponding to $\mathbf{i}$. C is parametrized as $y = 2cos\theta$, $z = 2sin\theta$, with $\theta \in [0, 2\pi]$. Along C, $x = 1$.

By Stokes' Theorem, we can compute the flux as following:

$$\begin{aligned}
the\ flux &= \int\int_S \mathbf{F} \cdot \mathbf{n}\ d\sigma \\
&= \int\int_S (\nabla \times \mathbf{G} \cdot \mathbf{n})\ d\sigma \\
&= \oint_C zdy - xdz \\
&= \int_0^{2\pi} [2sin(\theta)(-2sin(\theta)) - 2cos(\theta)]\ d\theta \\
&= \int_0^{2\pi} (-4sin^2(\theta) - 2cos(\theta))\ d\theta \\
&= -4\pi.
\end{aligned}$$

Problem 5 Find and classify all critical points of the function

$$f(x, y) = \frac{5}{2}x^2 - xy + 15x + \frac{1}{75}y^3 - 3y$$

Solution: Solve $\nabla f = (5x - y + 15)\mathbf{i} + (-x + \frac{1}{25}y^2 - 3)\mathbf{j} = \mathbf{0}$, one finds the critical points

$P_1(-3, 0)$ and $P_2(-2, 5)$.

To classify the type of the critical points, we perform the second derivative test.

$$f_{xx}(x,y) = 5,\ f_{xy}(x,y) = -1,\ f_{yy}(x,y) = \frac{2}{25}y.$$

For P_1, $A = 5$, $B = -1$, $C = 0$, $AC - B^2 = -1 < 0$, so P_1 is a saddle point.

For P_2, $A = 5$, $B = -1$, $C = \frac{2}{5}$, $AC - B^2 = 1 > 0$, so P_2 is a local minimum since $A > 0$.

Problem 6 True or False? Circle the correct answer. No partial credit.

- 1 : Any constant vector field $\mathbf{F}$ is a gradient field.

  (a) True (b) False.

  Solution: (a) True.

- 2: If C_1 and C_2 are two oriented curves, $\mathbf{F}$ is a vector field, and the length of C_1 is greater than the length of C_2, then $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} > \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$.

  (a) True (b) False.

  Solution: (b) False.

- 3: If $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F} = \mathbf{0}$.

  (a) True (b) False.

  Solution: (b) False.

- 4: If S_1 and S_2 are two oriented surface bounded by the same positively oriented curve C and $\mathbf{F}$ is a smooth vector field, the the flux of $\nabla \times \mathbf{F}$ through S_1 and S_2 are the same.

  (a) True (b) False.

  Solution: (a) True .

- 5 : If S is a unit sphere centered at the origin and $\mathbf{F}$ is a vector field that has zero total flux out of S, then $\nabla \cdot \mathbf{F} = 0$ at all points inside S.

  (a) True (b) False.

  Solution: (b) False.

Problem 7 Consider the surface S that is the part of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 3$.

(a) Give a parametric representation of S. Make sure to explicitly describe or sketch the parametrization domain D.

Solution: We can parametrize S by $\mathbf{r}(x, y) = rcos(\theta)\mathbf{i} + rsin(\theta)\mathbf{j} + r\mathbf{k}$, where (x, y) is inside the disc $x^2 + y^2 \leq 9$. Therefore, D is given by $0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 3$.

(b) Find an equation of the tangent plane to S at the point $P(-1, 1, \sqrt{2})$.

Solution: Let $g(x, y, z) = \sqrt{x^2 + y^2} - z$, S is the level surface of $g(x, y, z) = 0$.

$$\nabla g(-1,1,\sqrt{2}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - \mathbf{k}.$$

So the tangent plane to S at $P(-1,1,\sqrt{2})$ is

$$-\frac{1}{\sqrt{2}}(x+1) + \frac{1}{\sqrt{2}}(y-1) - (z-\sqrt{2}) = 0.$$

(c) If the density function $\lambda(x,y,z)$ is equal to the distance to the xy-plane, find the total mass of the surface S.

Solution: $\lambda(x,y) = z = r$. We compute the fundamental vector product

$$\mathbf{N}(r,\theta) = \mathbf{r}'_r \times \mathbf{r}'_\theta = -rcos(\theta)\mathbf{i} - rsin(\theta)\mathbf{j} + r\mathbf{k}$$

and thus $\|\mathbf{N}\| = r\sqrt{2}$.

$$\begin{aligned} M &= \int\int_S \lambda(x,y,z)\ d\sigma \\ &= \int_0^{2\pi}\int_0^3 r^2\sqrt{2}\ drd\theta \\ &= 18\sqrt{2}\pi. \end{aligned}$$

Remark: For this problem, one can also use the parametrization with a surface given by the graph $z = f(x,y) = \sqrt{x^2+y^2}$.

Problem 8 Let E denote the portion of the solid ball of radius R centered at the origin in the first octant, and let

$$\mathbf{F} = (2x + y)\mathbf{i} + y^2\mathbf{j} + cos(xy)\mathbf{k}.$$

Applying the Divergence Theorem, compute the net flux of the field $\mathbf{F}$ across the boundary of E, oriented by the outward-pointing normal vectors.

Solution: The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = 2 + 2y.$$

By the divergence theorem, the flux out of the given suface is equal to

$$\int\int\int_E (2 + 2y)dxdydz = 2(volum(E)) + 2\int\int\int_E ydxdydz,$$

where E is the region inside the surface. The volume of E is one eigth of the volume of the ball of radius R. Thus

$$2(volum(E)) = \frac{1}{3}\pi R^3.$$

In spherical coordinates, we have

$$
\begin{aligned}
2\int\int\int_E ydxdydz &= 2\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \rho sin(\theta)sin(\phi)\rho^2 sin(\phi)\ d\rho d\theta d\phi \\
&= \frac{R^4}{2}\int_0^{\pi/2}\int_0^{\pi/2} sin(\theta)sin^2(\phi)\ d\theta d\phi \\
&= \frac{R^4}{2}\int_0^{\pi/2} sin^2(\phi)\ d\phi \\
&= \tfrac{R^4}{8}\pi.
\end{aligned}
$$

So the final answer is

$$\frac{1}{3}\pi R^3 + \frac{1}{8}\pi R^4.$$

Problem 9 Please complete the course survey. Your comments will help me to improve my teaching in the future. Thank you in advance.