Problem 1. Calculations.

(a) Find the directional derivative of \( f(x, y, z) = xy + yz + zx \) at \( P(1, -1, 1) \) in the direction of \( i + 2j + k \)

Solution:
\[
\nabla f = (y + z)i + (x + z)j + (y + x)k,
\]
\[
\nabla f(1, -1, 1) = 2j. \quad u = \frac{\sqrt{6}}{6}(i + 2j + k), \text{ so}
\]
\[
f_u'(1, -1, 1) = \nabla f(1, -1, 1) \cdot u = \frac{2}{3} \sqrt{6}.
\]

(b) Find the rate of change of \( f(x, y) = xe^y + ye^{-x} \) along the curve \( r(t) = (\ln t)i + t(\ln t)j \).

Solution:
\[
\nabla f = (e^y - ye^{-x})i + (xe^y + e^{-x})j.
\]
\[
\nabla f(r(t)) = (t^t - \ln t)i + (t^t \ln t + \frac{1}{t})j,
\]
\[
\frac{df}{dt} = \nabla f(r(t)) \cdot r'(t) = t^t(\frac{1}{t} + \ln t + (\ln t)^2) + \frac{1}{t}.
\]

(c) Find \( \frac{\partial u}{\partial s} \) for \( u = x^2 - xy, x = scost, y = tsins \).
Solution:

\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} = (2x - y)(\cos t) + (-x)(\cos s) = 2\cos^2 t - t \sin t \cos s - st \cos s \cos t.
\]

(d) Find \( \frac{dy}{dx} \) if \( x\cos(xy) + y\cos(x) = 2 \).

Solution: Set \( u = x\cos(xy) + y\cos(x) - 2 \),

\[
\frac{\partial u}{\partial x} = \cos(xy) - y\sin(xy) - y\sin(x).
\]

\[
\frac{\partial u}{\partial y} = -x^2\sin(xy) + \cos(x).
\]

\[
\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\cos(xy) - y\sin(xy) - y\sin(x)}{x^2\sin(xy) - \cos(x)}.
\]

(e) Is \( \mathbf{F}(x, y) = (x + \sin y)\mathbf{i} + (x\cos y - 2y)\mathbf{j} \) a gradient of a function \( f(x, y) \)? If yes, find the general form of \( f(x, y) \).

Solution: Set \( P = x + \sin y, \quad Q = x\cos y - 2y \). \( \frac{\partial P}{\partial y} = \cos y = \frac{\partial Q}{\partial x} = \cos y. \)

Thus, \( \mathbf{F} \) is a gradient of a function. For \( f(x, y) \), we have from \( \frac{\partial f}{\partial x} = P \) that

\[ f(x, y) = \frac{1}{2}x^2 + xsin(y) + g(y). \]
To determine $g(y)$, we have

$$Q = \frac{\partial f}{\partial y} = x\cos(y) + g'(y),$$

which implies that $g'(y) = -2y$, thus $g(y) = -y^2 + C$, with $C$ a constant. So, $f(x, y) = \frac{1}{2}x^2 + x\sin(y) - y^2 + C$.

(f) Set $f(x, y) = \frac{x^2 - y^4}{x + y^4}$. Determine whether or not $f$ has a limit at $(1, 1)$.

**solution:** Along $x = 1$, the limit is 1, while along $y = 1$, the limit is $2/3$. So it has no limit at $(1, 1)$.

**Problem 2** Consider the function $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$.

(a) Find the equation for the tangent plane to the level surface $f = 4$ at the point $P(1, 4, 1)$.

**Solution:**

$$\nabla f = \frac{1}{2\sqrt{x}}i + \frac{1}{2\sqrt{y}}j + \frac{1}{2\sqrt{z}}k,$$

$$\nabla f(1, 4, 1) = \frac{1}{2}i + \frac{1}{4}j + \frac{1}{2}k.$$

Tangent plane: $\frac{1}{2}(x - 1) + \frac{1}{4}(y - 4) + \frac{1}{2}(z - 1) = 0$.

(b) Find the equation for the normal line to $f = 4$ at $P(1, 4, 1)$. 

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Solution: The normal line: \( x = 1 + \frac{1}{2}t, \ y = 4 + \frac{1}{4}t, \ z = 1 + \frac{1}{2}t. \)

(c) Use differentials to estimate \( f(0.9, 4.1, 1.1). \)

Solution: \( f(0.9, 4.1, 1.1) \approx f(1, 4, 1) + df. \)

\[
df = \frac{1}{2} \times (-0.1) + \frac{1}{4} \times 0.1 + \frac{1}{2} \times 0.1 = 0.025.
\]

Thus, the estimate is 40.025.

Problem 3. Find the area of the largest rectangle with edges parallel to the coordinate axes that can be inscribed in the ellipse \( \frac{x^2}{9} + \frac{y^2}{4} = 1. \)

Solution: Use Lagrange multiplies method. Set the coordinates of the corner points of the rectangle to be \((x, y), (-x, y), (-x, -y), (x, -y)\). We need to maximize \( f(x, y) = 4xy \) with the side condition \( g(x, y) = \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0. \)

\( \nabla f = 4yi + 4xj, \ \nabla g = \frac{2}{9}xi + \frac{1}{2}yj. \) Solve the following system:

\[
\begin{aligned}
4y = \lambda \frac{2}{9}x \\
4x = \lambda \frac{1}{2}y \\
g(x, y) = 0.
\end{aligned}
\]

We have

\( \lambda = 12, \ x = \frac{3}{2}\sqrt{2}, \ y = \sqrt{2}, \) and the area is 12.

Problem 4 Find the absolute extreme values taken on \( f(x, y) = \frac{-2y}{x^2+y^2+1} \) on the set \( D = \{(x, y) : x^2 + y^2 \leq 4\}. \)
Solution: \( \nabla f = \left( \frac{4xy}{(x^2+y^2+1)^2} i + \frac{2y^2-2x^2}{(x^2+y^2+1)^2} j \right) = 0 \) at \( P_1 = (0, 1) \) and \( P_2 = (0, -1) \) in \( D \).

Next we consider the boundary of \( D \). We parametrize the circle by

\[ C : r(t) = 2\cos(t)i + 2\sin(t)j, \ t \in [0, 2\pi]. \]

The values of \( f \) on the boundary are given by the function:

\[ F(t) = f(r(t)) = -\frac{4}{5}\sin(t), \ t \in [0, 2\pi]. \]

\( F'(t) = -\frac{4}{5}\cos(t) = 0 \) at \( t = \frac{1}{2}\pi \) and \( t = \frac{3}{2}\pi \). Thus the critical points on boundary of \( D \) are \( P_3 = r(0) = r(2\pi) = (2, 0), \ P_4 = r(\frac{1}{2}\pi) = (0, 2), \) and \( P_5 = r(\frac{3}{2}\pi) = (0, -2) \). Evaluate \( f \) at all critical points:

\[ f(0, 1) = -1, \ f(0, -1) = 1, \ f(2, 0) = 0, \]

\[ f(0, 2) = -\frac{4}{5}, \ f(0, -2) = \frac{4}{5}. \]

So, \( f \) takes on its absolute maximum of 1 at \((0, -1)\) and its absolute minimum of \(-1\) at \((0, 1)\).