Problem 1 Suppose that the matrix below is the augmented matrix of a system of linear equations

\[
\begin{pmatrix}
1 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & 3 & 2 \\
0 & -2 & 0 & 1 & 3 \\
0 & 0 & 1 & k & h
\end{pmatrix}
\]

a)(6 points) For what values of \(h\) and \(k\), this system has no solution.

Solution: By interchanging \(r_2\) and \(r_3\), and \(r_4 - r_3\), one arrives the REF of the matrix:

\[
\begin{pmatrix}
1 & 2 & 0 & 0 & 1 \\
0 & -2 & 0 & 1 & 3 \\
0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & k - 3 & h - 2
\end{pmatrix}
\]

Now, it’s easy to see that, the system has no solution if and only if the rightmost column is pivot. This happens if and only if \(k = 3\) and \(h \neq 2\).

b) (7 points) For what values of \(h\) and \(k\), this system has a unique solution. Find the solution.

Solution: Based on the REF derived from part a), the system has a unique solution if and only if \(k \neq 3\). In this case, the system is consistent without free variable. In order to solve the system, we row reduce the REF into RREF. This is achieved by \(\frac{1}{k-3}r_4\), \(-\frac{1}{2}r_2\), \(r_3 - 3r_4\), \(r_2 + \frac{1}{2}r_4\) and \(r_1 - 2r_2\). The RREF is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 4 - Y \\
0 & 1 & 0 & 0 & -\frac{3}{2} + \frac{1}{2}Y \\
0 & 0 & 1 & 0 & 2 - 3Y \\
0 & 0 & 0 & 1 & Y \\
\end{pmatrix},
\]

where \( Y = \frac{h - 2}{k - 3} \). So the solution is
\[x_1 = 4 - Y, \quad x_2 = -\frac{3}{2} + \frac{1}{2}Y, \quad x_3 = 2 - 3Y \quad \text{and} \quad x_4 = Y.\]

c)(7 points) For what values of \( h \) and \( k \), this system has infinitely many solutions. Describe the set of all solutions using parametric vector form.

**Solution:** From part a), the system has infinitely many solutions if and only if \( k = 3 \) and \( h = 2 \). In this case, the system is consistent with a free variable \( x_4 \). The REF is now
\[
\begin{pmatrix}
1 & 2 & 0 & 0 & 1 \\
0 & -2 & 0 & 1 & 3 \\
0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

To solve the system, we row reduce the above matrix to RREF by \( r_1 + r_2 \) and \(-\frac{1}{2}r_2\):
\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 4 \\
0 & 1 & 0 & -\frac{1}{2} & -\frac{3}{2} \\
0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Thus, \( x_1 = 4 - x_4, \quad x_2 = -\frac{3}{2} + \frac{1}{2}x_4, \quad x_3 = 2 - 3x_4 \) and \( x_4 \) is free. So the solution is described by
\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 
\end{pmatrix} = \begin{pmatrix} 4 \\ -\frac{3}{2} \\ 2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ \frac{1}{2} \\ -3 \\ 1 \end{pmatrix}.
\]
Problem 2 Let \( \mathbf{v} = (1, 0, 1)^t \). Define the linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) by \( T(\mathbf{x}) = \mathbf{v} \times \mathbf{x} \). Where

\[
\begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{pmatrix}
\times
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{pmatrix}
= 
\begin{pmatrix}
  a_2b_3 - a_3b_2 \\
  a_3b_1 - a_1b_3 \\
  a_1b_2 - a_2b_1
\end{pmatrix}.
\]

(a) Find the standard matrix \( A \) of \( T \).

Solution \( A = [a_1, a_2, a_3] \), where \( a_i = T(e_i) \).

\[
T(e_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad T(e_3) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.
\]

We thus have

\[
A = \begin{pmatrix}
  0 & -1 & 0 \\
  1 & 0 & -1 \\
  0 & 1 & 0
\end{pmatrix}.
\]

(b) Find a basis of \( \text{im}(A) \).

Solution: We do the interchange of \( r_1 \) and \( r_2 \), then \( r_3 + r_2 \), we thus reach the REF of \( A \):

\[
\begin{pmatrix}
  1 & 0 & -1 \\
  0 & -1 & 0 \\
  0 & 0 & 0
\end{pmatrix}.
\]

Therefore, we know that a basis of \( \text{im}(A) \) is \( \{a_1, a_2\} \).
c) What’s the dimension of $\ker(A)$?

**Solution** By the Rank Theorem, we know that

$$\dim \ker(A) = 3 - \dim \text{im}(A) = 1.$$  

**Problem 3** Consider an $m \times n$ matrix $A$ and an $n \times m$ matrix $B$ (with $n \neq m$) such that $AB = I_m$. Are the columns of $B$ linearly independent? What about columns of $A$?

**Solution:** If columns of $B$ are linearly dependent, so are columns of $AB$, which contradicts to $AB = I_m$. Or, we could show that columns of $B$ are linearly independent directly. To show this, we assume there is a vector $x \in \mathbb{R}^m$, such that $Bx = 0$. Then we have $x = I_m x = ABx = A0 = 0$. Thus, if $AB = I_m$, then columns of $B$ are linearly independent. Furthermore, we know that $n > m$. Since $A$ is $m \times n$, columns of $A$ are linearly dependent.

**Problem 4** Let $S = \{(x, y) : xy \geq 0\}$ be a subset of the plane $\mathbb{R}^2$. Is $S$ a subspace of $\mathbb{R}^2$?

**Solution:** $S$ is not a subspace of $\mathbb{R}^2$. One can easily verify that it is not close for addition. Choose $v = (-1, 0)$ and $u = (0, 1)$, both are in $S$, however, $v + u = (-1, 1)$ is not in $S$.

**Problem 5** For which values of the constant $k$ is the following matrix invertible? Find the inverse.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$$
**Solution:** Row reduce the matrix into REF by $r_2 - r_1$, $r_3 - r_1$, $r_3 - 3r_2$,

$$
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & k - 1 \\
0 & 0 & k^2 - 3k + 2
\end{pmatrix}.
$$

The matrix is invertible if $k^2 - 3k + 2 \neq 0$. Thus, if $k \neq 1$ and $k \neq 2$, the matrix is invertible. For $k \neq 1$ and $k \neq 2$, we denote the nonzero quantity $k^2 - 3k + 2$ by $N$, let $M = k - 1$, thus Gauss-Jordan algorithm will give the inverse

$$
\frac{1}{N} \begin{pmatrix}
2M + 2N - 2 & 3 - 3M - N & M - 1 \\
-N - 2M & N + 3M & -M \\
2 & -3 & 1
\end{pmatrix}.
$$