Problem 1 Suppose that the matrix below is the augmented matrix of a system of linear equations

\[
\begin{pmatrix}
1 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & 3 & 2 \\
0 & -2 & 0 & 1 & 3 \\
0 & 0 & 1 & k & h
\end{pmatrix}
\]

a) (6 points) For what values of $h$ and $k$, this system has no solution.

Solution: By interchanging $r_2$ and $r_3$, and $r_4 - r_3$, one arrives the REF of the matrix:

\[
\begin{pmatrix}
1 & 2 & 0 & 0 & 1 \\
0 & -2 & 0 & 1 & 3 \\
0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & k - 3 & h - 2
\end{pmatrix}
\]

Now, it's easy to see that, the system has no solution if and only if the rightmost column is pivot. This happens if and only if $k = 3$ and $h \neq 2$.

b) (7 points) For what values of $h$ and $k$, this system has a unique solution. Find the solution.

Solution: Based on the REF derived from part a), the system has a unique solution if and only if $k \neq 3$. In this case, the system is consistent without free variable. In order to solve the system, we row reduce the REF into RREF. This is achieved by $\frac{1}{k-3}r_4$, $-\frac{1}{2}r_2$, $r_3 - 3r_4$, $r_2 + \frac{1}{2}r_4$ and $r_1 - 2r_2$. The RREF is
where $Y = \frac{h-2}{k-3}$. So the solution is

$x_1 = 4 - Y$, $x_2 = -\frac{3}{2} + \frac{1}{2}Y$, $x_3 = 2 - 3Y$ and $x_4 = Y$.

c) (7 points) For what values of $h$ and $k$, this system has infinitely many solutions. Describe the set of all solutions using parametric vector form.

**Solution:** From part a), the system has infinitely many solutions if and only if $k = 3$ and $h = 2$. In this case, the system is consistent with a free variable $x_4$. The REF is now

\[
\begin{pmatrix}
1 & 2 & 0 & 0 & 1 \\
0 & -2 & 0 & 1 & 3 \\
0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

To solve the system, we row reduce the above matrix to RREF by $r_1 + r_2$ and $-\frac{1}{2}r_2$:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 4 \\
0 & 1 & 0 & -\frac{1}{2} & -\frac{3}{2} \\
0 & 0 & 1 & 3 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Thus, $x_1 = 4 - x_4$, $x_2 = -\frac{3}{2} + \frac{1}{2}x_4$, $x_3 = 2 - 3x_4$ and $x_4$ is free. So the solution is described by

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = \begin{pmatrix}
4 \\
-\frac{3}{2} \\
2 \\
0
\end{pmatrix} + x_4 \begin{pmatrix}
-1 \\
\frac{1}{2} \\
-3 \\
1
\end{pmatrix}.
\]
Problem 2 Let $v = (1, 0, 1)^t$. Define the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(x) = v \times x$. Where

\[
\begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3 
\end{pmatrix}
\times
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 
\end{pmatrix} =
\begin{pmatrix}
  a_2b_3 - a_3b_2 \\
  a_3b_1 - a_1b_3 \\
  a_1b_2 - a_2b_1 
\end{pmatrix}.
\]

a) Find the standard matrix $A$ of $T$.

Solution $A = [a_1, a_2, a_3]$, where $a_i = T(e_i)$.

\[
T(e_1) = \begin{pmatrix}
  0 \\
  1 \\
  0 
\end{pmatrix},
T(e_2) = \begin{pmatrix}
  -1 \\
  0 \\
  1 
\end{pmatrix},
T(e_3) = \begin{pmatrix}
  0 \\
  -1 \\
  0 
\end{pmatrix}.
\]

We thus have

\[
A = \begin{pmatrix}
  0 & -1 & 0 \\
  1 & 0 & -1 \\
  0 & 1 & 0 
\end{pmatrix}.
\]

b) Find a basis of $im(A)$.

Solution: We do the interchange of $r_1$ and $r_2$, then $r_3 + r_2$, we thus reach the REF of $A$:

\[
\begin{pmatrix}
  1 & 0 & -1 \\
  0 & -1 & 0 \\
  0 & 0 & 0 
\end{pmatrix}.
\]

Therefore, we know that a basis of $im(A)$ is $\{a_1, a_2\}$. 

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c) What’s the dimension of $\ker(A)$?

**Solution** By the Rank Theorem, we know that

$$\dim \ker(A) = 3 - \dim \operatorname{im}(A) = 1.$$ 

**Problem 3** Consider an $m \times n$ matrix $A$ and an $n \times m$ matrix $B$ (with $n \neq m$) such that $AB = I_m$. Are the columns of $B$ linearly independent? What about columns of $A$?

**Solution:** If columns of $B$ are linearly dependent, so are columns of $AB$. Thus, if $AB = I_m$, then columns of $B$ are linearly independent. Furthermore, we know that $n > m$. Since $A$ is $m \times n$, columns of $A$ are linearly dependent.

**Problem 4** Let $S = \{(x, y) : xy \geq 0\}$ be a subset of the plane $\mathbb{R}^2$. Is $S$ a subspace of $\mathbb{R}^2$?

**Solution:** $S$ is not a subspace of $\mathbb{R}^2$. One can easily verify that it is not close for addition. Choose $v = (-1, 0)$ and $u = (0, 1)$, both are in $S$, however, $v + u = (-1, 1)$ is not in $S$.

**Problem 5** For which values of the constant $k$ is the following matrix invertible? Find the inverse.

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & k \\
1 & 4 & k^2 \\
\end{pmatrix}
$$

**Solution:** Row reduce the matrix into REF by $r_2 - r_1$, $r_3 - r_1$, $r_3 - 3r_2$, 

4
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & k - 1 \\
0 & 0 & k^2 - 3k + 2
\end{pmatrix}.
\]

The matrix is invertible if \( k^2 - 3k + 2 \neq 0 \). Thus, if \( k \neq 1 \), or 2, the matrix is invertible. For \( k \neq 1 \) and \( k \neq 2 \), we denote the nonzero quantity \( k^2 - 3k + 2 \) by \( N \), let \( M = k - 1 \), thus Gauss-Jordan algorithm will give the inverse

\[
\frac{1}{N} \begin{pmatrix}
2M + 2N - 2 & 3 - 3M - N & M - 1 \\
-N - 2M & N + 3M & -M \\
2 & -3 & 1
\end{pmatrix}.
\]